Carry Trade and Return Crash Risk*

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Abstract

A model is developed in order to show that in the carry trade market the Sharpe ratio can be affected by the number of traders and it has a concave form. Hence, the Sharpe ratio does not increase with the interest rate differential. However, high interest rate currencies have greater currency crash risk exposure. The exchange rate movement, when there is no currency crash, does not affect so much the profit due to the carry trade, but the total profit is very sensitive to the exchange rate fluctuation. Skewness and Kurtosis are computed for 9 currencies as indexes for currency crash risk and Sharpe ratio is calculated as a proxy for profitability. In the empirical part, the Sharpe ratio shows a concave form and the model predict this concavity too. The model captures the effect of number of arbitrageurs in the carry trade market on the profit. In the last section, the exchange rate risk premium in this market is computed.

Keywords: carry trade, crash risk, exchange rate risk premium, Sharpe ratio

JEL Classification: E44, F31, G12

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1 Introduction

A currency carry trade is defined as a leveraged cross-currency position to take advantage from interest rate differentials and high Sharpe ratio of this market. The strategy consist of borrowing fund from banks in a low interest rate currency and investing it in a currency with higher interest rate. The former currency is often called funding currency and the latter one target currency. Notice that the strategy is profitable for an unhedged carry trade strategy, when the interest rate differential is high enough to compensate exchange rate fluctuations and so the uncovered interest parity is not expected to hold. We will see in the results that the profit of carry trade is very sensitive to exchange rate movement. In fact traders put on the carry trades whenever they think the UIP will not hold. That means the exchange rate movement will not necessarily offset those differences between countries’ interest rates. We show this exchange rate risk premium in the last section of the paper. In this definition buying foreign assets, though is involved with unhedged currency exposure, is not classified as carry trade.

Probably the most interesting feature of carry trade for speculators is the leveraged position of carry trade through violation of the uncovered interest parity (UIP). The UIP states that the currency with higher return (interest rate) will depreciate against the currency with lower return in order to have the condition of risk-free arbitrage. Therefore under UIP the profit through interest rates differential is offset by the exchange rate movement. In fact this is true for a long-term period, but carry trade strategies consist of investing in short-term periods. Chinn and Meredith (2006) confirm that UIP holds for periods longer than 5 years. In fact what happens in reality is inverse of what UIP predicts. Many papers discussed that currencies with high interest rates tend to appreciate while other currencies with low interest rates depreciate. This is called the Forward Premium Puzzle. Furthermore some researchers suggest that forward premium puzzle is not necessarily a pre-condition to put on carry trade, but it can be the consequence of carry trade. The idea is introduced by Froot and Thaler (1990). They tested unbiasedness of exchange rate changes provided by interest rate differential with this equation:

$$\Delta s_{t+k} = \alpha + \beta(i_t - i_t^*) + \nu_{t+k}$$

where $\Delta s_{t+k}$ is the exchange rate change over $k$ periods, $(i_t - i_t^*)$ is the term of interest rates differential and $\nu$ is the term of error. The null hypothesis is $\beta = 1$. The result is that $\beta$ is less than one and indeed it is negative. The negative $\beta$ means that the money which offers higher interest rate tends to appreciate. The authors thought a possible explanation could be, the slow response of some participants to interest rates differential changes. The failure of UIP is known to market
participants and therefore some of them have created tradable benchmarks and they have introduced FX instrument referencing these benchmarks. Gyntelberg and Remolona (2007) mentioned some of these indexes in their paper. Among them all, the one from Deutsche Bank, DB HARVEST, refers more to Asian currencies and its method of calculation is simpler (not necessarily better).

Carry trade does not tend to involve major currencies, instead it involves currencies with high return such as Australian dollar, New Zealand dollar, Island krona, Turkish Lira, Brazilian real, Hungarian forint and even occasionally pound sterling. Carry trade involves mostly Japanese yen and Swiss franc and sometimes US dollar as the funding currencies. However the situation of the US dollar is different since 2004. The dollar served as a potential target currency rather than a funding currency until 2007 and then with a decrease in the US interest rate it became a funding currency again.

The following figures show different interest rates classified by target and funding currency.

![Funding Currencies](image)

**Figure 1: Funding Currencies.**

Japanese Yen and Swiss Franc are supposed to be the funding currencies because they have the lowest interest rate among other currencies of our sample.
US dollar is supposed to be an occasionally target currency. In fact this was true until recently. Now the US interest rate is practically zero and it can be regarded as a new funding currency. The interest rate in the Euro zone has almost similar pattern, with the difference that its volatility is lower than the US interest rate.

This group of currencies are the popular destination for the carry traders, since these currencies have relatively stable economies (except Iceland recently) relative to the developing countries and they offer high interest rates.
The last group of currencies offer very high interest rates but they have a high volatility both in the interest rate and in the exchange rate and therefore the carry trade expose to a higher return crash.

The rest of the paper is organized as follow: In the next section, we review the related literature to our work. In section 2, we will describe the data and draw some stylized facts about the carry trade market. Section 3 presents a model to rationalize the stylized facts discussed in section 2. Finally, the last section introduce the exchange rate risk premium and try to discover its relationship with the carry trade and then we conclude.

2 Literature Review

Brunnermeier et al (2008) showed that carry trades are subject to currency crash risk, i.e., the exchange rate movements of carry trade portfolios are negatively skewed. The authors argue that this skewness in foreign exchange rates follows from temporary changes in the availability of funding liquidity to arbitrageurs. When the funding liquidity is temporarily reduced, this results in a rapid unwinding of the traders’ positions and thus leads to abrupt changes in the exchange rates, which go against the carry traders. This risk, they argue, is a major factor affecting traders’ willingness to enter into these risk arbitrage positions and arbitrage away the positive returns to carry trades. Brunnermeier and Pederson (2008) look at the relationship between funding liquidity and asset market liquidity, but in a general context. In their model they documented that market liquidity can suddenly dry up. Adrian et al (2009) showed that foreign exchange
markets are influenced by liquidity conditions. They used balance sheets of financial intermediaries as a tool to predict the future returns on the currency market. They found that liquidity channel is related to the exchange rate risk premium and this is associated to the carry trade incentives.

Burnside et al (2008) refer to the peso problem as an explanation for the high average payoff to the carry trade. Peso problem is defined as a generic term for the effects of small probabilities of large events on empirical work. Their approach relies on analyzing the payoffs to a version of the carry-trade strategy that does not yield high negative payoffs in a peso state. This strategy works as follows. When an investor sells the foreign currency forward he simultaneously buys a call option on that currency. If the foreign currency appreciates beyond the strike price, the investor can buy the foreign currency at the strike price and deliver the currency in fulfillment of the forward contract. Similarly, when an investor buys the foreign currency forward, he can hedge the downside risk by buying a put option on the foreign currency. By construction, this hedged carry trade does not generate large negative payoffs in the peso state. To estimate the average payoffs to the hedged carry trade the authors used data on currency options with a one-month maturity. At this stage of the analysis Burnside et al wish to be eclectic about the size of the negative payoffs in the peso state. So, their hedging strategy uses at-the-money options which pay off in all peso states, as well as in some non-peso states. The main results of the paper are as follow: first the average payoffs to the hedged and unhedged carry trade are very similar. Second the standard deviation of the payoffs to the hedged carry trade is actually substantially lower than those of the unhedged carry trade. Third the payoffs to the unhedged carry trade in the peso state is only moderately negative and finally the SDF is over one-hundred times larger in the peso state than in the non-peso state. Fahri et al (2009) decompose the profit in the carry trade market into the profit due to a Gaussian risk premium and the profit due to a disaster risk premium. They link the disaster risk premium to the impact of disasters on SDF and the carry trade payoffs in disaster periods. They argue that with a non-hedged carry trade, we can only compute Gaussian risk premium and this strategy does not allow to capture disaster risk premium. In order to compute this risk premium, they uses currency options. After estimation, they showed that disaster risk explain about 30 percent of the carry trade returns. From this point of view their strategy of hedging is very similar to those of Burnside et al. It is worth mentioning that the Peso state in the former article corresponds the disaster risk. Jurek (2008) investigates whether the excess return in the carry trade market is due to the exposure to currency crash. For this purpose, he used the dynamics of the moments of the risk-neutral distribution implied by the currency options. Also he examined the return to the carry trade market in which the risk of currency crash has been
hedged by currency options (as the previous papers). The results shows that crash risk premium explains 30-40% of the total excess return to currency carry trades which is similar to the paper of Farhi et al. However taking into account the risk in the carry trade market, the excess return still remains very high. Bhansali (2007) build a hedged portfolio in the carry trade market with using the currency options. Bhansali shows that the volatility of such option is proportional to the currencies interest rates differential. The author shows theoretical and empirical supporting evidences for positive connection between volatility and carry.

Plantin and Shin (2008) explore the carry trade market using asset pricing framework. They show that without funding costs and with the known and constant fundamental value of asset, speculation is not possible on the market and therefore, the presence of carry costs change completely the previous results. By introducing carry costs and allowing the fundamental value to be stochastic, they generate a speculative dynamic in this market. According to the authors, these results highlight the importance of the interaction between carry costs and the sensitivity of prices to flows. The main goal of is paper is to show that in the carry trade market, bubble can exist. They paid attention to the phenomena of going up by the stairs and down by the elevator for the currencies involved in the carry trade market. Many studies on bubbles paid attention to this phenomena. Abreu and Brunnermeier (2003) and Veldkamp (2004) report this slow boom and sudden crash in financial markets. Veldkamp explains this pattern with an endogenous flow of information that varies with the economic activity level. This information mechanism endogenously generates unconditional asymmetry in lending rate and investment changes. She introduces two measures for judging the asymmetry of data: time-irreversibility and skewness.

Jylhä et al (2008) model the carry trade market with two countries whose habitants are risk-averse and due to the high transaction cost they can only invest in their own country's securities markets. Besides there exists an Island on which a limited number of risk-averse arbitrageurs are living and they can buy and sell short fixed income securities in all markets. The model contains inflation risk which is different across the countries. Each country is endowed with risky and riskless assets and the economy has only two periods. All investors have CARA utility function with respect to their wealth in the second period. The authors show that in absence of arbitrageurs the equilibrium interest rate is higher in the country with the higher inflation risk. When the arbitrageurs are present in the market they maximize their utility by selecting the number of shares of the risky asset purchased in country with higher interest rate financed by selling shares of the risky asset in country with lower interest rate. Jylhä et al show that taking into account the presence of arbitrageurs the same result holds. In the empirical
part of the paper the authors estimate that carry trade is a major strategy for the hedge funds (which their assets under management is about 6% of the M2 money supply of major currencies) and so it has affected the currency market and return to the carry trade. The spirit of our model is relatively similar to their model.

3 Data and Some Stylized facts

3.1 Some Stylized facts in Carry Trade Market

Datastream is used for the exchange rates and interest rates. The exchange rates are in daily basis for each country and we used the average for the monthly, quarterly and yearly data. We also used interest rates in the monthly, quarterly and yearly basis. In this section, we document some stylized facts in carry trade market with a descriptive and econometric approach. The bellowing figure is the dollar exchange rate against the yen.

![USD/JPY](image)

Figure 5: USD/JPY exchange rate.

Brunnermeier et al (2008) paid attention to the relationship between dramatic exchange rate movement and carry trade. As it can be seen, there was a dollar crash in October 1998. This risk (Peso state or disaster risk in other literature) can cause a big loss for traders in the carry trade market. In fact, we can decompose the profit of carry trade to the profit due to the interest rate differentials and the profit (or loss) due to the target (funding depreciation) exchange rate appreciation (depreciation). The authors found this drastic depreciation to have no relationship with fundamental news announcement but it can more be related due to the unwinding of carry trade. This is an example of going up by the stairs and down
by the elevator. Here, we use skewness and Kurtosis as measures to calculate asymmetry of exchange rates changes.

### 3.2 Risk in the Carry Trade Market: A descriptive approach

In this section, Skewness and Kurtosis are used as measures of exchange rate risk. Skewness is used to show the risk of currency crash while Kurtosis measures whether these crashes are abrupt or not (We use these measures to compute the Gaussian risk in the Farhi et al paper). A big negative skewness means that the exchange rate has been appreciated slowly and is crashed suddenly while a big positive Kurtosis shows that this crash is fast. Kurtosis is also a measure of whether the data are peaked or flat relative to a normal distribution. That is, data sets with high kurtosis tend to have a distinct peak near the mean, decline rather rapidly, and have heavy tails. Data sets with low kurtosis tend to have a flat peak near the mean rather than a sharp peak. A uniform distribution would be the extreme case. The skewness for a normal distribution is zero, and any symmetric data should have a skewness near zero. Negative values for the skewness indicate data that are skewed left and positive values for the skewness indicate data that are skewed right. By skewed left, we mean that the left tail is long relative to the right tail. Similarly, skewed right means that the right tail is long relative to the left tail. Also positive kurtosis indicates a peaked distribution and negative kurtosis indicates a flat distribution. Figure (11) shows skewness and Kurtosis for some target currencies. Almost all pair of currencies have positive big Kurtosis and negative skewness. Negative skewness of in this context confirms the slow boom of exchange rates and sudden crash. Big positive kurtosis confirms that these changes are fast. The most extreme case in this figure is in (11d). Also the measures in panel (11c) capture very well the high depreciation of the US dollar against the yen in October 1998. In the last quarter of 1998 the skewness of USD/JPY is -2.35 and its kurtosis attains to 10.91.

#### 3.2.1 Currency crash risk

As discussed above, the exchange rate movement is not symmetric when it goes up and when it falls down. This is investigated through skewness and kurtosis. Skewness and kurtosis for every currency is plotted and showed in the appendix. The asymmetry of exchange rate movement is associated with a crash risk. The price of this risk (or the risk premium) is labeled as risk reversal price. Risk reversal is a long position in an out-of-the-money call option combined with a short position of an equally out-of-the-money put, that is, call-put. Risk reversal is in fact an option to restrict the loss while it limits the maximum profit. Under
risk neutral measure, exchange rate is symmetrically distributed and so the price of risk reversal is zero, since the value of buying a call offset the value of selling a put. But for example if the exchange rate is negatively skewed (data are skewed left) the price of risk reversal is negative.

The skewness and Kurtosis are calculated of daily exchange rate changes within each month and each quarter since July 1996. Figures (12) and (13) depict skewness and kurtosis quarterly for all 9 currencies against Japanese yen. These figure shows clearly that all currencies are skewed negatively relative to Japanese yen. Panel 13j is JPY/TKL (whereas TKL/JPY is so small it was better to simulate JPY/TKL) exchange rate and it shows that the yen is positively skewed against Turkish lira, therefore TKL is negatively skewed. Negative skewness confirm the argument going up by the stairs and down in the elevator. The argument is true even for currencies like euro and dollar which are targeted less comparing to other currencies. Negative skewness is the risk for the speculators and they can assure themselves via buying risk-reversal. Almost in all panels it can be seen that whenever skewness has a negative peak, the corresponding kurtosis has a positive peak, which means the changes occurred very rapidly. This happens especially in figure (13) which contains currencies with high interest rates.

Figure (6) shows the average skewness and the kurtosis for all currencies since July 1996 until July 2008 versus interest rate differentials. Although average skewness and average kurtosis does not show the exact situation, they are useful to understand the general tendency in the carry trade market. This figure shows where the interest rate is higher, the skewness is more negative. Also figure (7) for the average kurtosis demonstrates that higher interest rate currencies have higher kurtosis.
Euro has the highest skewness (in term of absolute value) and the least kurtosis. Also Brazilian real offers highest interest rate and therefore has the highest
kurtosis and the least skewness. This pattern is true for other currencies too. More the return is, more the risk (negative skewness and positive kurtosis) is. Therefore the figure describes very well the risk associated to currency crash. Maybe we can divide currencies into three groups: currencies with low return and low risk (euro and pound), currencies with medium return and higher risk comparing to the first group (USD, AUD, NZD) and finally the last group which contains high-return high-risk currencies (ISK, HUF, BRL). We can estimate from similarity of interest rate movement that Turkey belongs to the last group. This classification is consistent with the figures showed in the introduction for the exchange rate co-movements.

3.2.2 Concavity of the Sharpe ratio in the Carry trade market

Figure (8) shows the Sharpe ratio versus interest rate differentials which is calculated for different currencies. We will show in the theoretical part that the Sharpe ratio has a concave form. Here the maximum takes place for the Hungarian forint. The figure shows that the excess return due to the high interest rate differentials is compensated by the high risk of currency crash for Brazil Real and Turkey Lira. We will interpret more deeply this figure in the theoretical part.

Figure 8: Sharpe ratio vs IR differentials 1996-2008.
3.3 The International Common Shocks

In this subsection \(^1\), we document some empirical stylized facts. The goal is to identify international transmission of shocks. To do so, we used Panel Vector AutoRegression (PVAR) framework. The most general form of the model can be written as:

\[
\Pi_{i,t} = \mu_i + \Theta(L)\Pi_{i,t-1} + \epsilon_{i,t}, i = 1, ..., N, t = 1, ..., T
\]

where \(\Pi_{i,t}\) contains six variables (Skewness, Kurtosis, Exchange rate, Profit, Interest rate differential). \(\mu_i\) is the country idiosyncratic effect, \(\epsilon_{i,t}\) is the residual error and \(\Theta(L)\) is a lag operator with \(\Theta(L) = \Theta_1L + \Theta_2L^2 + ....... + \Theta_pL^p\).

In order to compute impulse respond functions (IFR), we identify the shocks using Choleski decomposition. This decomposition introduce some restrictions on contemporaneous correlations between variables. PVAR methodology is also useful to take into account the endogeneity problem and the inter-relationship dynamic between the variables. Helmert transformation is used in order to remove endogenously the individual effects. The variables are sorted by the most exogenous to most endogenous.

A third-order PVAR have been estimated with these variables, using monthly data from July 1996 until July 2008. Some results are presented in the following figures.

![Figure 9: Response of Profit to Interest rate differentials shock](image)

This figure shows the response of profit to a positive shock to the interest rate differentials. This means that interest rate differentials have a relatively big positive impact on the payoff.

\(^1\)We used mainly the STATA codes developed by Inessa Love. Financial Development and Dynamic Investment Behavior: evidence form Panel VAR (with Lea Ziccino), The Quarterly Review of Economics and Finance, 46(2) (May 2006), 190-210.
This figure confirms econometrically the previous graphs which were plotted for different currencies. It shows a positive shock to skewness has a negative effect on profit. In other words, the higher the risk of currency crash is, the higher is the probability of return crash risk.

In the following sections, we will rationalize these stylized facts.

4 The Model

4.1 Setup of the model

We have three countries, the first is the supplier of the funding currency and the second is the investment destination target currency and $K$ arbitrageurs in a third country. We suppose that arbitrageurs take a long position in the funding currency and a short position in the target currency. The final profit is expressed in the currency of the arbitrageur. At the end of the investment period, the arbitrageur get her money back from the target country and must repay the borrowed money from funding currency. So, she is exposed to two exchange rate risks: the depreciation of the target currency and/or the appreciation of the funding currency. Her profit comes from the interest rates differentials between the target and funding currency adjusted by the exchange rate movements.

For the moment, the dynamic of the exogenous exchange rate is assume as follow:

$$dS_t = S_t (\mu dt + \sigma dw_t)$$

where $\mu$ and $\sigma$ are the drift and the volatility of the exchange rate and $w_t$ is a Brownian motion. Using Ito’s lemma, it’s easy to show that the level of the
exchange rate can be written as:

\[ S_t = S_0 e^{\left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \omega_t} \]

So, \( \frac{S_{t+1}}{S_t} \) follows a lognormal distribution with a mean of \( \left( \mu - \frac{\sigma^2}{2} \right) \) and a variance of \( \sigma^2 \).

4.2 Dynamic of wealth and Utility function

At time \( t + 1 \), the wealth of the representative arbitrageur \( k \) is:

\[ W_{t+1}^k = \bar{W}_t^k + \left( \frac{S_{t+1}^+}{S_t^+} \right) \lambda^+ R^+ - \left( \frac{S_{t+1}^-}{S_t^-} \right) \lambda^- R^- \]

+ index is for the target currency and - is for the funding currency. \( R^+ = 1 + r^+ \) is the interest rate of the target currency and \( R^- = 1 + r^- \) as the interest rate of the funding currency. \( S_t \) is the exchange rate which is the amount of the foreign currency for one unit of arbitrageurs' currency. \( \lambda^+ \) and \( \lambda^- \) are the shares that the arbitrageurs use for the short and long positions in their portfolios. The first term is her initial wealth which can be assumed exogenous and constant, and the second term is the profit due to the short position in the target currency and the last one is the long position in the funding currency.

We impose the following condition in the strategy of the arbitrageur:

\[ \lambda^+ S^+ = \lambda^- S^- \]

This constraint means that the amount of money that the arbitrageur borrows from the funding currency should be equal to the amount of money invested in the target currency.

If arbitrageurs are risk-neutral, they do care only about their expected payoffs and not the risk in the carry trade market. Hence, the risk averse arbitrageurs allows us to build a trade off between payoff and risk. So, arbitrageurs maximize the following utility function:

\[ U(W_{t+1}^k) = -e^{-\gamma_k W_{t+1}^k} \]

where \( \gamma_k \) is the risk aversion parameter.

4.3 Solving the model

The maximization problem is:

\[ \max_{\lambda^+, \lambda^-} E \left( -e^{-\gamma_k W_{t+1}^k} \right) \]
\[ \lambda^+ S^+ = \lambda^- S^- \]

The maximization problem for \( \lambda^+ \) and \( \lambda^- \) under the constraint yields:

\[
\lambda^+ = \frac{R^+ \Gamma^+ - \left( \frac{S^+_t}{S^-_t} \right) R^- \Gamma^-}{\gamma_k \left[ R^+ \Sigma^2_{S^+} + R^- \Sigma^2_{S^-} \left( \frac{S^+_t}{S^-_t} \right)^2 - 2R^+ R^- \left( \frac{S^+_t}{S^-_t} \right) (\Omega - \Gamma^+ \Gamma^-) \right]} \tag{1}
\]

\[
\lambda^- = \frac{\left( \frac{S^-_t}{S^+_t} \right) R^+ \Gamma^+ - R^- \Gamma^-}{\gamma_k \left[ R^+ \Sigma^2_{S^+} \left( \frac{S^-_t}{S^+_t} \right)^2 + R^- \Sigma^2_{S^-} - 2R^+ R^- \left( \frac{S^-_t}{S^+_t} \right) (\Omega - \Gamma^+ \Gamma^-) \right]} \tag{2}
\]

where \( \Gamma \) is the expected value of depreciation (appreciation) of each currency and \( \Sigma^2_S \) is its variance. \( \Omega \) is a constant which is expressed in the appendix. The nominators of these fractions are the adjusted return to the exchange rate movement and the denominators are risk factors. These equations express that the amount of money in long position should increase when the interest rates differential goes up and should decrease whenever the risk of currency crash is high. The risk of currency, here, is measured by the volatility of the exchange rates, i.e., \( \Sigma^2_S \). The relative risk aversion parameter is appeared in the denominator, which is consistent with the intuition. The last term in the denominators is the covariance between the target and the funding currency and since we assumed that the dynamic of the both currencies is affected by the same Brownian process, this term appears with a negative sign in the risk term (denominator here) and if the two stochastic processes are not correlated this term vanishes.

According to these equations, an arbitrageur puts in the carry trade market if he expects a positive return. This result is similar to Jylhä at al (2008).

**Lemma 1** Arbitrageur borrows in the funding currency and lends in the target currency, if and only if the expected returns from carry trade is strictly positive i.e., \( R^+ \Gamma^+ > \left( \frac{S^+_t}{S^-_t} \right) R^- \Gamma^- \).
4.4 The impact of the number of arbitrageurs

The same technique is used in order to maximize the profit of trader in the funding and target currencies. The dynamic of wealth for a funding currency trader is:

\[ W_{t+1}^- = \bar{W}_t^- + R^- \lambda_F^- \frac{S_t^-}{S_{t+1}^-} \]

\( \bar{W}_t^- \) is assumed exogenous and constant. The same form of utility function is applied for this traders. Therefore, the maximization problem is:

\[ \max_{\lambda_F^-} \mathbb{E} \left( -e^{-\gamma^- W_{t+1}^-} \right) \]

With \( \Gamma_F = e^{-\mu^- + \sigma^-} \), the maximization problem yields

\[ \Rightarrow \lambda_F^- R^- = \frac{\Gamma_F}{\gamma^- \Sigma^-} \]

The latter equation helps us to simplify the calculations furthermore by substituting the \( \lambda_F^- R^- \) with a constant. Next we verify the market clearing condition between the arbitrageurs and the funding currency supplier. We assume that we have \( K \) arbitrageurs and one supplier. Therefore the market clearing condition is:

\[ \sum_{k=1}^{K} \lambda_k^- = \lambda_F^- \]

Suppose that all arbitrageurs have the same risk aversion parameter, \( \frac{1}{\gamma_k} = \rho = \text{const} \), and defining \( \alpha = \lambda^- \gamma_k \) we get:

\[ \alpha \rho K = \lambda_F^- \]

\[ \lim_{K \to \infty} \alpha = 0 \]

which means that the nominator of equation (1) should tend to zero. In other words, we have

\[ \frac{S_t^+}{S_t^-} R^+ \Gamma^+ = R^- \Gamma^- \]

Lemma 2 When the number of traders tend to infinity, the arbitrage opportunity vanishes.

This is the UIP condition in absolute term value. In fact, since we see big profit on carry trade market, the above lemma means that the carry trade market is characterized by a non-competitive structure. This can be due to entering barriers, excessive collateral, asymmetric information or higher trading costs.
4.5 Sharpe Ratio

In this section, the goal is to compute the Sharpe ratio and see how it varies with different parameters of the model. The Sharpe ratio is a measure of the excess return per unit of risk. It is defined as:

\[
SR = \frac{\mathbb{E} \left[ \lambda^+ \frac{S_{t+1}^+}{S_t^+} R^+ - \lambda^- \frac{S_{t+1}^-}{S_t^-} R^- \right]}{\text{VAR} \left( \lambda^+ \frac{S_{t+1}^+}{S_t^+} R^+ - \lambda^- \frac{S_{t+1}^-}{S_t^-} R^- \right)^{\frac{1}{2}}}
\]

With some further manipulation, we have

\[
SR = \left( \frac{S^-}{S^+} \right) R^+ \Gamma^+ - R^- \Gamma^-
\]

\[
\left( (R^+)^2 \left( \frac{S^-}{S^+} \right)^2 \Sigma^+ + (R^-)^2 \Sigma^- - 2 (R^+ R^-) \left( \frac{S^-}{S^+} \right) [\Omega - \Gamma^+ \Gamma^-] \right)^{\frac{1}{2}}
\]

Using equation (1) we are able to rewrite the Sharpe ratio:

\[
(SR)^2 = \gamma_k \lambda \left[ \left( \frac{S^-}{S^+} \right) R^+ \Gamma^+ - R^- \Gamma^- \right]
\]

The lemma when the number of arbitrageurs goes to infinity the arbitrage opportunity disappears. Here we can see it again in term of the Sharpe ratio. When \(k \to \infty\), \( \left( \frac{S^-}{S^+} \right) R^+ \Gamma^+ = R^- \Gamma^- \) and consequently the Sharpe ratio tends to zero.

**Lemma 3** When the number of the traders tend to infinity, the Sharpe ratio tend to zero.

This lemma means the more competitive is the carry trade market, the less reasonable is to choose a riskier action in order to get a bigger profit. The profit must disappear whatever the level of risk is. In other words, when the number of trader increases, the market becomes more competitive and as microeconomic textbooks show, the profit must vanish whatever the level of risk is.

4.6 Concavity of the Sharpe ratio

In this part, we want to show that the Sharpe ratio has some maximum relative to the interest rates differentials. Otherwise, if the interest rate differentials are bigger than some thresholds, the Sharpe ratio decreases. This is maybe due to the crash risk of the currencies with the high interest rate. Thus, we want to confirm the fact that is showed in the figure (8). For this purpose, we have to calculate the first and the second derivatives
of the Sharpe ratio with respect to the interest rates differential. We had from the market clearing condition:

\[ K\lambda^- = \lambda_F^- \]

multiplying by \( R^- \)

\[ \lambda^- R^- = \frac{\Gamma_F}{\gamma^- K\Sigma^-} \]

Substituting this in the Sharpe ratio relation (Eq. 3):

\[ SR^2 = \frac{\gamma_k}{\gamma^-} \frac{\Gamma_F}{K\Sigma^-} \left[ \left( \frac{S^-}{S^+} \right) \left( \frac{R^+}{R^-} \right) \Gamma^+ - \Gamma^- \right] \]

Let’s rewrite the Sharpe ratio equation by using the following notation: \( R = \frac{R^+}{R^-} \) and \( S_t = \frac{S^-}{S^+} \).

\[ SR^2_t = \frac{(S_t R\Gamma^+ - \Gamma^-)^2}{R^2 S_t^2 \Sigma^2_s + \Sigma^2_s - 2 R S_t [\Omega - \Gamma^+ \Gamma^-]} \]

Now we are able to use this formula to calculate the maximum of the Sharpe ratio.

\[ 2(SR)_k \frac{\partial(SR)_k}{\partial R} = 0 \]

This gives an equation of 4th degree. To see the concavity of the Sharpe ratio, we derive equation (4). The first derivative is

\[ 2(SR)_k \frac{\partial(SR)_k}{\partial R} = \frac{\gamma_k}{\gamma^-} \frac{\Gamma_F}{K\Sigma^-} \left( \frac{S^-}{S^+} \right) \Gamma^+ \]

The rhs of the above equation is constant and therefore the second derivative can be calculated as follow:

\[ \frac{\partial^2(SR)_k}{\partial R^2} = -\frac{1}{(SR)_k} \left( \frac{\partial(SR)_k}{\partial R} \right)^2 < 0 \]

This second derivative is negative if and only if

\[ (SR)_k > 0 \Leftrightarrow \frac{S^-}{S^+} R^+ \Gamma^+ > R^- \Gamma^- \]

which means that the interest rate of the target currency adjusted to the exchange rate movement should be higher than the interest rate of the funding currency. Using Lemma (1), we know that this is always true for an arbitrageur who wants enter into this market. Thus the second derivative is negative and the Sharpe ratio has a concave form respect to the interest rates differentials.

**Lemma 4** In the carry trade market, the Sharpe ratio has a concave form with respect to the interest rates differentials.

As it is showed in figure (8), excess returns due to the high interest rate differentials will be compensated by the high risk of currency crash. The concave form of the Sharpe ratio captures this property.
5 Exchange Rate Risk Premium

The risk in the carry trade market is the exchange rate movement. According to the non-arbitrage condition in the risk-neutral world the profit should be equal to zero. This condition can be written formally as:

$$E_t \left[ M_{t+1} \left( \lambda^+ R^+ \frac{S_{t+1}^+}{S_t^+} - \lambda^- R^- \frac{S_{t+1}^-}{S_t^-} \right) \right] = 0$$

(5)

where $M$ is the stochastic discount factor.

The UIP condition can be written in the logarithm term as:

$$s_{t+1} - s_t = r^+ - r^- + \mu + \epsilon_{t+1}$$

where $s_t = \frac{S_t^+}{S_t^-}$, so the rhs is the exchange rate appreciation (depreciation), $\mu$ is the risk premium and $\epsilon_{t+1}$ is an idiosyncratic risk which follows a normal distribution with a zero mean and a variance equal to 1. In fact, we have the following formula for $\mu$:

$$\exp(\mu) = \phi = \frac{1}{E(M_{t+1})} \left[ \lambda^+ R^+ Cov \left( M_{t+1}, \frac{S_{t+1}^-}{S_t^-} \right) - \lambda^+ R^+ Cov \left( M_{t+1}, \frac{S_{t+1}^+}{S_t^+} \right) \right]$$

To calculate the risk premium we proceed as follow:

$$E_t \left[ M_{t+1} \left( \lambda^+ R^+ \frac{S_{t+1}^+}{S_t^+} \right) \right] = 1$$

$$E_t \left[ M_{t+1} \left( \lambda^- R^- \frac{S_{t+1}^-}{S_t^-} \right) \right] = 1$$

Assuming log-normality of the pricing kernel and also for the term in the parentheses in the above equations and taking the logarithms of them gives:

$$E_t \left[ m_{t+1} + \epsilon^+ + r^+ + \Delta s_{t+1}^+ \right] + \frac{1}{2} V_t \left[ m_{t+1} + \epsilon^+ + r^+ + \Delta s_{t+1}^+ \right] = 0$$

$$E_t \left[ m_{t+1} + \epsilon^- + r^- + \Delta s_{t+1}^- \right] + \frac{1}{2} V_t \left[ m_{t+1} + \epsilon^- + r^- + \Delta s_{t+1}^- \right] = 0$$

Choosing $w_{t+1}^\pm = \epsilon^\pm + (1+r^\pm) + \Delta s_{t+1}^\pm$ as the logarithm of the the first and the second term of wealth, $\epsilon^\pm = \ln \lambda^\pm$ and $m_{t+1} = \ln M_{t+1}$ and subtracting the second equation from the first equation gives:

$$E_t(w_{t+1}^+-w_{t+1}^-) - \frac{1}{2} V_t(w_{t+1}^+-w_{t+1}^-) = - Cov_t \left( w_{t+1}^+-w_{t+1}^-, m_{t+1} + \epsilon^+ + (1+r^+) + \Delta s_{t+1}^+ \right)$$

Therefore the risk premium can be obtained by the following equation:

$$\mu = E_t(w_{t+1}^+-w_{t+1}^-) - \frac{1}{2} V_t(w_{t+1}^+-w_{t+1}^-) + Cov_t \left( w_{t+1}^+-w_{t+1}^-, m_{t+1} \right) + Cov_t \left( w_{t+1}^+-w_{t+1}^-, \Delta s_{t+1}^\pm \right)$$
Simplifying more and using the equation in the appendix, \( \mu \) can be written as:

\[
\mu = -\left(\frac{1}{2} V_t (\Delta s_{t+1}^+) - \frac{1}{2} V_t (\Delta s_{t+1}^-)\right) - \text{Cov}_t (\Delta s_{t+1}^+, \Delta s_{t+1}^-, m_{t+1})
\]

The term in the parentheses is the Jensen effect and of the second order which arises because expectations are being taken of a nonlinear function. The second term is the dominant term in measuring the risk premium. If we look at the SDF as the marginal utility of the traders, the above equation means that, in this market the risk premium (or expected excess return) depends mainly on the relationship between the exchange rate movement and traders’ preferences. There are many methods to measure the covariance term. One of the simplest is for example to assume a CIR process for the SDF which contains only one factor. From empirical point of view, modeling the SDF correspond to take into account some risk factors of economy and hence its covariance with the exchange rate depreciation (appreciation) would be meaningful. Therefore, to do so, we should first, endogenize the exchange rate process and then we will be able to calculate the covariance. However this is left for future researches.

6 Conclusion

We tried to construct a portfolio for a trader in a third party country. The traders’ strategy is unhedged in this research. In other words, traders do not immune themselves against target currency depreciation or funding currency appreciation and by buying some forward contracts.

Using a Panel VAR, it is shown that the profit of doing carry trade is very sensible to the exchange rate movement and interest rate differentials profit. Notice that although the exchange rate profit is negligible and even negative, it affects the total profit so much. Skewness and Kurtosis are depicted for all currency pairs. All currencies are negatively skewed relative to the Japanese yen and their kurtosis is positive. This is the case even for the currencies (as euro and pound) that normally are not used for carry trade. Negative skewness means that currencies are disposed to a crash risk. Average skewness and kurtosis is plotted against interest rates differentials. The figure shows clearly that higher interest rate currencies have more negative skewness and more positive kurtosis. This means higher return currencies are riskier. The 9 currencies can be divided into three sub groups according to their return and their exchange rate risk exposure. The exchange rates of currencies in each group have a co-movement with each other.

Finally excess profit and Sharpe ratio are computed. Sharpe ratio is a measure of profitability. Generally speaking the carry trade market offers a very high Sharpe ratio comparing to the other markets. First Sharpe ratio increases for the high return currencies but after a while it decreases. The concavity of the Sharpe ratio is shown by the model too. Thus choosing between currencies to invest depends on how much traders are risk reversal.

In the theoretical part, the model shows that the arbitrageur put in the market, whenever they expect that the UIP does not hold. Next, we showed that when the number of trader goes to infinity, the carry trade opportunity vanishes and it looks like a
competitive market. In term of the Sharpe ratio, this is interpreted as non-profitability and therefore the Sharpe ratio tends to zero. The important contribution of the model is to show that the Sharpe ratio is concave.

In the last section, we computed the exchange rate risk premium. We showed that the risk premium is equal to the covariance between the exchange rate movement and the pricing kernel adjusted by the Jensen effect.
References


APPENDIX

The maximization problem

In this part, the notations and calculations are showed in detail. Using the dynamic of the profit, we can write:

\[ E \left( -e^{-\gamma_k W_{t+1}^i} \right) = -E \exp \left[ -\gamma_k \lambda^+ \frac{S_{t+1}^+}{S_t^+} R^+ + \gamma_k \lambda^- \frac{S_{t+1}^-}{S_t^-} R^- \right] \]

We use the following notation for the mean and variance of the exchange rate:

\[ E \left( \frac{S_{t+1}}{S_t} \right) = e^{(\mu - \frac{\sigma^2}{2})} \exp(\sigma(W_{t+1} - W_t)) \]
\[ = e^{(\mu - \frac{\sigma^2}{2})} e^{\frac{\sigma^2}{2}} = e^\mu = \Gamma \]

and for the variance

\[ \text{VAR} \left( \frac{S_{t+1}}{S_t} \right) = \text{VAR} \left( e^{(\mu - \frac{\sigma^2}{2})} \exp(\sigma(W_{t+1} - W_t)) \right) \]
\[ = e^{(2\mu - \sigma^2)} \left[ \exp(2\sigma(W_{t+1} - W_t)) - \left( \exp(\sigma(W_{t+1} - W_t)) \right)^2 \right] \]
\[ = e^{(2\mu - \sigma^2)} \left[ e^{2\sigma^2 - \sigma^2} \right] = e^{2\mu} \left( e^{\sigma^2} - 1 \right) \]
\[ = \Sigma_S^2 \]

then, we can write

\[ E \left[ -\gamma_k \lambda^+ \frac{S_{t+1}^+}{S_t^+} R^+ + \gamma_k \lambda^- \frac{S_{t+1}^-}{S_t^-} R^- \right] = -\gamma_k \lambda^+ R^+ \Gamma^+ + \gamma_k \lambda^- R^- \Gamma^- \]

and

\[ \text{VAR} \left[ -\gamma_k \lambda^+ \frac{S_{t+1}^+}{S_t^+} R^+ + \gamma_k \lambda^- \frac{S_{t+1}^-}{S_t^-} R^- \right] = \gamma_k^2 \left( \lambda^+ R^+ \right)^2 \Sigma_S^2 + \left( \lambda^- R^- \right)^2 \Sigma_S^2 - 2 \left( \lambda^+ \lambda^- \right) \left( R^+ R^- \right) \left[ \Omega - \Gamma^+ \Gamma^- \right] \]

with

\[ \Omega = e^{(\mu^+ + \mu^- + \sigma^+ + \sigma^-)} \]

and the expected value of the utility function is:

\[ \max_{\lambda^+}, \lambda^- \quad -\exp \left[ -\gamma_k \left( \lambda^+ R^+ \Gamma^+ - \lambda^- R^- \Gamma^- \right) + \frac{\gamma_k^2}{2} \left( \lambda^+ R^+ \right)^2 \Sigma_S^2 + \left( \lambda^- R^- \right)^2 \Sigma_S^2 - 2 \left( \lambda^+ \lambda^- \right) \left( R^+ R^- \right) \left[ \Omega - \Gamma^+ \Gamma^- \right] \right] \]
Skewness and Kurtosis

In the following figures skewness and kurtosis of all currencies versus Japanese yen are shown.

Figure 11: Skewness and Kurtosis for some target currencies
Figure 12: Skewness and Kurtosis quarterly-Exchange rate vs Japanese yen
Figure 13: Skewness and Kurtosis quarterly-Exchange rate vs Japanese yen-high return currencies