Copula-Use in Interest Rate Risk Management of Russian Banks[1]

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ABSTRACT

The paper is aimed at comparing the efficiency of copula models and traditional approaches to interest rates management of a Russian bank. Copula application enabled to reveal that the Russian interest rates joint multivariate distribution is asymmetric, i.e. interest rates tend more frequently to rise simultaneously than to decline. It is also shown that copulas enable to diminish the number of expected value of equity-at-risk breaches by 7-13% depending on the chosen confidence level compared to GARCH approach.

1. Purpose of the study

Bad debt crisis originated in the USA and consequently expanded to economies all over the world threatened capital adequacy of most commercial banks. Numerous cases of unpaid loans write-offs were reported as a loss for the current period and, thus, decreased the capital of credit organisations. For the purpose of maintenance of capital adequacy level, governments and central banks in the USA, European Union and Russia have decided to provide subordinated loans that are legally permitted to be treated as banks' equity[3].

The aim of this study is to elaborate the optimal model of interest rate risk estimation that would permit to better forecast the change in the expected present

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[3] Refer to Statute 215-П of the Central Bank of the Russian Federation (paragraph 3.5.2) and [BCBS (2006)], paragraph 49(xii) (the latter source stands for the capital adequacy framework for credit organisations, Basel II).

value of a bank's capital by the current approaches due to capturing non-Gaussian nature of multivariate risk distribution.

The paper is organised as follows. Section 2 describes different approaches to measuring interest rate risk exposure with a focus on methods of forecasting of yield curve changes which are the principal interest rate risk-factor. Section 3 explains the methodology of the study. Section 4 encompasses the data on which the econometric modelling was based. The results of the study are summarised in section 5. Section 6 concludes the paper.

2. Approaches to measuring the interest rate risk exposure

Interest rate risk is one of the types of market risk and is defined as expected losses from interest rate changes as a result of gaps between banks' assets and liabilities. This risk, as well as other bank risks, is also a source of potential gains. Otherwise in the absence of gaps (i. e. under perfectly balanced term structure of a bank's balance sheet) a bank would have a low profit margin between active and passive operations that would not be sufficient even for covering operational costs.

For the purpose of interest rate risk management two types of gaps are considered: liquidity gap and repricing gap. The former allows to measure the interest rate risk emanating from bank's net assets, the latter – the risk relating to interest income. Also, depending on the maturity of assets and liabilities, gaps can be classified as either contractual (settled in the agreement) or behavioural, or expected contractual (suggesting that the expected maturity of an asset or liability is estimated).

Currently there are two main approaches to measuring interest rate risk exposure in accordance with the international agreement on capital adequacy measurement (Basel II): standardised and internal models approach.

It is worth noting that under the legal framework[4] the interest rate risk exposure should be determined only for trading portfolio of fixed-income securities (bonds). However, interest rate changes affect the present value of all assets and liabilities. Therefore, this study focuses on the gap analysis of all assets and liabilities rather than solely of that of the trading portfolio.

(a) Standardised approach

The simplest techniques for measuring a bank's interest rate risk exposure is a standardised approach. It consists in distributing assets and liabilities (as it was mentioned earlier, for a trading portfolio that suggests distinguishing between long and short positions in securities, respectively) according either to their maturities or duration[5]. Exogenously each time bucket is given a risk weight depending on the bond's coupon rate. Such weights reflect potential losses resulting from expected interest rate movements for a corresponding maturity.

As it follows from the table 1 below [please refer to BCBS (2006), paragraph 718 (iv)] , risk weights increase as the maturity extends that is the result of the following functional dependence on which the notion of duration is based :

$$\Delta P = -D \cdot \Delta r \qquad (1)$$

Therefore, the greater the maturity of an asset (liability) is, the more significant change in the present value is caused by the same interest rate movement.

[4] The Bank of Russia requires that the market (and therefore, interest rate) risk exposure be calculated if only « the total present (fair) values of financial instruments is no less than 5% of total assets of a credit organisation » [Statute 313-П, paragraph 1.3.1], while the US Federal Reserve System obliges to measure the market risk exposure (equal to the total amount of trading assets and liabilities, for legal purposes) when it exceeds 10% of total assets or $1 bn. [USGAO (1998), p. 121].

[5] Duration reflects the sensitivity of a bond price change to a change in interest rates, i.e. in continuous time $D = -\partial P / \partial r$ or in discrete time $D = -\Delta P / \Delta r$, where D is duration, P is a bond price, r is a general interest rate level in the economy. In spite of the fact that duration measures the interest rate elasticity of a bond price it is often used to measure the security's maturity.

Table 1. Risk weights distribution by maturity

Coupon rate of 3% and more	Coupon rate of less than 3%	Risk weight	Expected change in returns
1 month and less	1 month and less	0,00%	1,00
1 to 3 months	1 to 3 months	0,20%	1,00
3 to 6 months	3 to 6 months	0,40%	1,00
6 to 12 months	6 to 12 months	0,70%	1,00
1 to 2 years	1 to 1,9 years	1,25%	0,90
2 to 3 years	1,9 to 2,8 years	1,75%	0,80
3 to 4 years	2,8 to 3,6 years	2,25%	0,75
4 to 5 years	3,6 to 4,3 years	2,75%	0,75
5 to 7 years	4,3 to 5,7 years	3,25%	0,70
7 to 10 years	5,7 to 7,3 years	3,75%	0,65
10 to 15 years	7,3 to 9,3 years	4,50%	0,60
15 to 20 years	9,3 to 10,6 years	5,25%	0,60
More than 20 years	10,6 to 12 years	6,00%	0,60
	12 to 20 years	8,00%	0,60
	More than 20 years	12,50%	0,60

Statute 313-П of the Central Bank of the Russian Federation introduces rigid risk weights for measuring interest rate risk exposure. These weights are similar to those suggested by Basel II Accord for coupon rates of 3% and more (refer to the table above). However, this approach does not account for a simultaneous movement in risk factors (i.e. interest rates).

(b) Internal Models Approach

In order to design a more flexible risk management framework and to solve the above mentioned drawbacks, Basel II allows credit organisations to use their internal models subject to the local Central Bank's approval.

As noted in the study [Prefontaine, Desrochers (2006)], the main internal banks' model of interest rate risk exposure measurement is Expected Value of Equity-At-Risk – EVEaR over a given time horizon and at a determined confidence level. The calculation principle consists in the estimation of the present value of net assets at current and expected interest rates. Consequently, the value at risk is determined as a quantile of a given level α of the distribution of the gap between these present values.

Thus we would like to stress that interest rate risk management is based on forecasting the yield curve. Therefore, let us consider available approaches to its forecasting in more detail.

An important step in forecasting the yield curve was made by Nelson and Siegel [Nelson, Siegel (1987)] who suggested the parametrisation of the yield curve in the form of distinguishing three components (short-, medium- and long-term) for the purpose of forecasting.

This approach was applied to the data on the Russian market for the first time by Gambarov et al. for forecasting the government bond (OFZ) yield curve [Gambarov et al. (2004)], [Gambarov et al. (2006)].

We would also like to cite the article [Chegotov, Lobanov (2006)] that describes an interesting approach to yield curve forecasting. The authors use prices of coupon government bonds OFZ in Nelson-Siegel one-factor model that implies using prices of zero-coupon bonds as input data. Prices of six OFZ issues from December 3, 2003 to August 31, 2006 and overnight MIACR[6] are analysed.

The resulting yield curve on average corresponds to observed values. Yet the authors note the problem of "breaches" due to peak MIACR values since those rates effectively include corporate risk premium in addition to sovereign risk (refer to table 6 below). The Kalman filtration is also applied in order to solve this issue.

Despite the transparency and the simplicity of parametric models approach currently the most actively used models of yield curve forecasting are affine models with embedded Kalman filter.

The development of the first group of models was started by Vasicek [Vasicek (1977)]. It was described in more detail later by Duffie and Kan [Duffie, Kan (1996)] and Dai and Singleton [Dai, Singleton (1998)].

[6] MIACR stands for **M**oscow **I**nterbank **AC**tual **R**ate.

Thus the paper [Duffie, Kan (1996)] introduces the form of the affine model where the return Y_{it} at a time t for the maturity τ_i is determined by the value of the vector of factors X_t by the following formula:

$$Y_{it} = -\frac{A(\tau_i) + B(\tau_i)X_t}{\tau_i}, \qquad (2)$$

where $A(\tau_i)$, $B(\tau_i)$ are the coefficients that are a function of the maturity τ_i and that are subject to the following border conditions: $A(0) = 0$ and $B(0) = 0$. These conditions are derived from the fact that if affine dependence $\alpha + \beta \cdot x = 0$ holds for all x in any non-empty open Euclid space then $\alpha = 0$ and $\beta = 0$[refer Duffie, Kan (1996), p. 383].

No-arbitrage condition is also introduced when $A(\tau_i) = B_j(\tau_i), \forall i \neq j$.

The authors note that there are two approaches to choosing the factors X : explicit use of yield curve observations and implicit modifications. Given the complications associated with calibrating the covariance matrix to empirical data in the latter variant the authors opt for the former approach.

Also the paper indicates that although the choice of the overnight rate as a factor is traditional it is not a necessary condition since it is impossible to observe the effective instant rate.

All these papers assume the symmetric movement of interest rates. In other words the hypothesis about the multivariate normal joint distribution of changes in interest rates is accepted. Nonetheless, other researchers [Junker, Szimayer, Wagner (2003)] refused this assumption and attempted to forecast the yield curve by applying copula for modelling innovations, i. e. noise component in the time series model.

Their study is based on monthly returns of U.S. treasury bonds for the period from 1982 to 2001.

The authors consider the affine model of term structure of interest rates with two factors : returns on 1-year and 2-year bonds. Copula is used for generating the relevant residuals in the model by Monte-Carlo method . In order to build the yield curve under the affine model framework the Kalman's filter is also used, although

Kalman filter deficiencies are discussed because its base assumption about the normal distribution of original innovations does not hold.

For testing the accuracy of resulting models seven indicators are used : Anderson-Darling test, Kolmogorov-Smirnov test, AIC, BIC, relative deviation of the density of upper and lower tails of distribution and general χ^2-test.

The results of the study show that reverse Frank's copula is the best to fit empirical data. The authors underline that the results can be used for estimating VaR as a quantile of the distribution of returns of a bond portfolio.

It is worth noting that copulas started to be actively used in developed countries for risk management purposes since they allow not only to separate modelling of individual (marginal) distributions and the type of dependence (see section « Definition, properties and types of copulas » for further details), but also to model non-elliptical multivariate distributions. Below we provide a succinct description of advantages of copula-based approaches and the drawbacks of traditional approaches (e. g. using linear correlation as a dependence measure). Then we resume the studies which focused on copula-modelling and the conclusions about the choice of optimal copula depending on the aim of the study and the type of data used.

The authors Ane and Kharoubi [Ane, Kharoubi (2003)] point out the drawbacks of the assumption about the normal joint distribution of the returns for financial time series and using linear correlation for measuring the interdependence of random variables.

In the beginning the researchers consider four copula families : Gaussian, Frank's, Gumbel's, Cook-Johnson. These copulas are estimated on the basis of empirical data by maximum likelihood method using the Silverman's kernel estimate for marginal distributions. The result of estimation is compared to the empirical Dehivel's copula. The optimal copula is defined by means of three criteria: simple and integral Anderson-Darling test and entropy measure.

The authors show that Cook-Johnson copula better describes empirical data since it captures a non-zero correlation of negative observations ([Longin, Solnik (1998)] showed that correlation between returns is stronger on the bearish (descending) than on bullish (ascending) market).

The researchers base their study on daily quotes of six indices (FTSE100, DAX30, Nikkey 225, Hang Seng, S&P500 и NASDAQ) for the period from January 2, 1987 to December 31, 2000.

The paper shows that the assumption about the Gaussian joint distribution of returns provides an optimistic estimate of risk level in terms of Value-at-Risk, thus underestimating the risk. The paper also contains the calculation results for loss function that illustrates that the choice of the type of marginal distribution explains 80% of estimation accuracy whereas the choice of interdependence structure (copula) explains the rest 20%.

The paper [Clemen, Reilly (1999)] primarily focuses on the role of correlation in building the joint distribution. The first part of the study analyses the approaches to estimating the interdependence between random variables while the second part calibrates the resulting estimates based on the values of the original correlation matrix.

In the first part of the paper the authors consider random variables X and Y with marginal distribution functions $F_X(x)$ and $G_Y(y)$, respectively. The coherence of expert opinions about three types of dependence listed below is analysed:

1) Linear correlation;
2) Coherence probability $P(X > x_1; Y > y_1)$; and
3) Conditional fractiles estimate

$$E(F_X \mid y) = r(X,Y) \cdot (G_Y(y) - 0.5) + 0.5, \qquad (3)$$

where $r(X,Y)$ is the coefficient of correlation between two random variables.

Respondents were offered three pairs of stock indices from U.S. database Compustat in comparison with their calculated values. Respondents were shown scatter plots and other descriptive statistics of marginal rather than joint distributions. Despite the fact that people who took part in the poll were inclined to underestimate the values of the first two types of dependence the authors are reluctant in choosing any of these three types for using it as a copula parameter.

The second part of the study analyses a hypothetical four-factor asset pricing model that was designed through a multivariate normal copula. The first calibration method consists in transforming the entire correlation matrix from the case of total independence between random variables (zero correlation coefficients) to the case of total positive dependence (coefficients equal to 1). The second method consists in changing only non-zero values for +/- 0,25% (that corresponds to the average deviation of correlation from calculated values, after respondents' responses). There is no significant difference reported in calibration results. General sensitivity varied from -50% to +100% of value, i.e. calibration gave minimum and maximum values of VaR equal to $6552 and $22049 whereas average calculated value was $12417 (before calibration).

The paper [Embrechts, McNeil, Straumann (1999)] describes problems that may arise when using linear correlation as the only measure of interdependence of random variables in case of non-elliptical distributions.

For capturing interdependencies corresponding to outcomes observed in reality, the structure of dependence should be modelled in addition to correlation. This task is successfully solved by copula modelling. In particular, the authors provide an example of generation of two joint distributions with marginals in the form of gamma (3,1) and linear correlation coefficient (0,7), but different copulas: Gaussian (left) and Gumbel's (right).

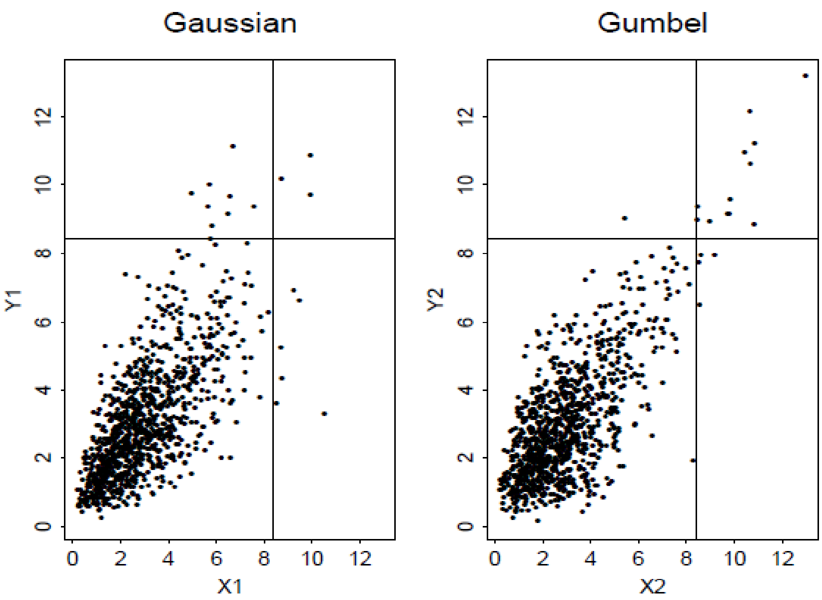


Figure 1. For Gumbel's copula the distribution is characterised by greater extremums than for Gaussian copula.

The paper comprehensively describes correlation types (linear, rank, tail dependence) in a static world, i.e. series correlation of time series is not taken into consideration. The preference should be given to rank correlation that relates quantiles of the distribution of random variables rather than their outcomes. The article summarises the requirements for dependence measuring. The issues of distribution modelling and imitating under the assumption of different copulas are discussed.

Below we provide the summary of the main applied research papers that show (1) that applying copulas allows better capturing the structure of a multidimensional distribution when marginal distributions are not Gaussian ; (2) the majority of papers reveal an asymmetric (non-elliptical) structure of the joint multivariate distribution of financial data.

Table 2. Previous evidence on copula-modelling of joint distributions.

Study	Data type	Observation period (frequency)	Considered copulas	The best copula
Ane, Kharoubi, 2003	Equity indices	1987 – 2000 (daily)	Gaussian Frank's Gumbel's Clayton's	Clayton's
Chollete, Heinen, 2006	Equity indices	1990 – 2002 (weekly)	Gaussian	Mix of Gaussian
Junker, Szimayer, Wagner, 2003	US T-bills quotes	1982 – 2001 (monthly)	Gaussian Student's Gumbel's Frank's (reverse)	Frank's (reverse)
Fantazzini, 2009	Equity indices	1994 – 2000 (daily)	Normal Student's	Normal
Hsu, Tseng, Wang, 2007	Equity indices	1995 – 2005	Gaussian Clayton's Gumbel's	Gumbel's (for cross hedge)[7] Normal (for direct hedge)
Kole, Koedijk, Verbeek, 2006	Equity indices, bond indices, real estate indices	1999 – 2004 (daily)	Gaussian Student's Gumbel's	Student's
Morone, Cornaglia, Mignola, 2007	Banking risks	2002 – 2005 (monthly)	Normal Student's	Student's
Patton, 2006	Exchange rates	1991 – 2001 (daily)	Gaussian Clayton's	Clayton's
Rosenberg, Schuermann, 2004	Banking risks	1994 – 2002 (quarterly)	Normal Student's	Student's
Tang, Valdez, 2006	Insurance risks	1992 – 2002 (semi-annual)	Normal Student's Cochi	Cochi
Fantazzini, 2008	Share prices	2004 – 2008	Normal Student's	Student's

[7] Cross hedge is the act of hedging one's position by taking an offsetting position in another asset with similar price movements. Direct hedge implies taking the offsetting position in the same asset. For instance, for direct hedging of a futures contract for oil supply a reverse contract (that is normally non-deliverable) for oil is signed, whereas in a cross hedge this supply is offset by a contract on gas the price of which is highly correlated with oil price.

Study	Data type	Observation period (frequency)	Considered copulas	The best copula
Alexeev, Shokolov, Solozhentsev, 2006	Share prices	2002 – 2004 (daily)	Ali-Mickael-Khak Clayton	Ali-Mickael-Khak

3. Methodology of measuring interest rate risk exposure

(1) Definition, properties and types of copulas

Copula is a strictly increasing function of several variables defined on an n-dimensional cube $[0;1]^n$ with values on $[0;1]$.

$C:[0;1]^n \to [0;1]$, i.e. C is a transformation converting a point of an n-dimensional hypercube in a point on [0;1].

Copulas (from latin *couple*) are used for construction of general multidimensional distribution $F_X(x)$ based on the knowledge of marginal distributions $F_{X_i}(x_i)$. This result rests on the following **Sklar's theorem** [Sklar (1959)].

> *Let H be a joint distribution function of two random variables (x, y) with marginal distribution functions F and G, respectively. Then there exists a copula C such that for all x, y* $(-\infty;+\infty)$ *it holds*
>
> $$\exists C : H(x,y) = C[F(x);G(y)], \forall x, y \in (-\infty;+\infty) \qquad (4)$$
>
> *If functions F and G are continuous then the copula C is unique; otherwise the copula C can always be defined on the interval of values of F and G. Conversely, if C is a copula and F and G are marginal distribution functions, than the above defined function H is a joint distribution function of F and G.*

Therefore, as it follows from the definition, for any joint n-dimensional distribution there exists a distribution function in the form of a copula C defined by the following relation:

$$F(X) = C[F_1(x_1);...;F_n(x_n)] \qquad (5)$$

Consequently, the copula density distribution function takes the following form :

$$c(x_1,...,x_n) = \frac{\partial^n C(x_1,...,x_n)}{\partial x_1 \cdot ... \cdot \partial x_n} \qquad (6)$$

Below are the main **properties of copulas** (two-dimensional case) :

1. Copula is grounded;
   - C(u,0)=C(0,v)=0 (7)
   - C(u,1)=u; C(1,v)=v (8)
2. Any copula lies in Frechet-Hoeffding boundaries;
   - $Max(0, u+v-1) \le C(u,v) \le Min(u,v)$ (9)
3. If random variables under consideration are statistically independent then holds C(u,v)=uv ;
4. If U and V are perfectly linearly correlated, then C(u,v)=Min(u,v) ;

Copula types

The table below summarises distribution functions of copulas used in this study.

Copula name **(generator for archimedean[8] copulas)**	**Copula formula** $C(x_1,...,x_n)$	
Gaussian	$\int_{-\infty}^{x_n} ... \int_{-\infty}^{x_1} \frac{1}{\sqrt{(2\pi)^n}\,\lvert\Sigma\rvert} \exp\left(-\frac{1}{2} z^T \Sigma^{-1} z\right) \partial z_1 ... \partial z_n$, where z_i is the distribution function[9];	(10)

[8] Nelson [Nelson (2006), pp. 115,122] describes the origins of the Archimedean copula's notion first introduced by Ling in 1965[Ling (1965)].

[9] It is necessary to note that copulas of elliptical distributions do not require the marginal distribution to be symmetric.

Copula name **(generator for archimedean[8] copulas)**	**Copula formula** $C(x_1,...,x_n)$	
Student's[10]	$\dfrac{\Gamma\left(\dfrac{v+n}{2}\right)}{\Gamma\left(\dfrac{v}{2}\right)(v\pi)^{n/2}\sqrt{\lvert\Sigma\rvert}}\int_{-\infty}^{x_n}...\int_{-\infty}^{x_1}\left(1+\dfrac{1}{v}z^T\Sigma^{-1}z\right)^{-(v+n)/2}\partial z_1...\partial z_n$, where z_i is the distribution function; v is the number of copula's degrees of freedom	(11)
Frank's[11] $-\dfrac{1}{\alpha}\log\left[1+e^{\alpha t}\left(e^{-\alpha}-1\right)\right]$	$-\dfrac{1}{\alpha}\log\left[1+\dfrac{\prod_{i=1}^{n}\left(e^{-\alpha x_i}-1\right)}{\left(e^{-\alpha}-1\right)^{n-1}}\right]$	(12)
Clayton's[12] $(t+1)^{-1/\alpha}$	$\left[\sum_{i=1}^{n}x_i^{-\alpha}-n+1\right]^{-1/\alpha}$	(13)
Gumbel's[13] $\exp(-t^{1/\alpha})$	$\exp\left\{-\left[\sum_{i=1}^{n}(-\log x_i)^{\alpha}\right]^{1/\alpha}\right\}$	(14)

(2) **Methodology of measuring interest rate risk exposure**

This study consists of three key stages. The first step was to estimate conditional heteroscedasticity models for yield curve forecasting. The second step

[10] Student's copula with one degree of freedom is called Cochi's copula, similar to the distribution, and refers to the category of copulas with extreme values since it is characterised by a high value of tail dependence parameter.

[11] This type of liason of multivariate distributions was first suggested in the paper [Frank (1979)]. Frank's copulas can be estimated both in ordinary logarithm [Tang, Valdez (2006), p.5-6] and natural logarithm [Nelson (2006), pp. 116].

[12] The original idea of this type of copula was first introduced in the paper [Clayton (1979)] and then described in more detail in the article [Cook, Johnson (1981)]. Therefore, this copula is often called both Clayton's copula and Cook-Johnson copula [Tang, Valdez (2006), p. 6].

[13] The idea of this form of interdependence, which was later called Gumbel's copula, was first introduced in the paper [Gumbel (1960)].

was to find the optimal copula describing the distribution of residuals of the estimated model. The third step was to compute the retrospective forecast EVEaR based on the calculated forecast values of the yield curve.

In spite of a wide popularity of affine models with embedded Kalman filter (filtration approach was first suggested in [Kalman (1960)]), we consider that it is more important to account for volatility waves. Choudhry and Wu [Choudhry, Wu (2007)] also make a conclusion about the higher accuracy of forecasts for GARCH model compared to Kalman filter.

The paper [Penikas (2008)] shows that using combined forecasts allows increasing the average accuracy of forecasts for GARCH model. Nonetheless, this approach still does not allow capturing the asymmetric structure of the joint distribution of residuals in the model. Therefore for a full time series we have estimated seven copulas (Gaussian, Clayton's, Gumbel's, Frank's and Student's with one, three and ten degrees of freedom) by maximum likelihood techniques. For each copula we have calculated the copula parameter as well as the tail dependence index for upper (λ_U) and lower (λ_L) tails that are defined by the following formula in a two-dimensional case [Nelson (2006), p. 214].

$$\lambda_U = \lim_{t \to 100^-} P\left[Y > G^{(-1)}(t) \mid X > F^{(-1)}(t) \right] \tag{15}$$

$$\lambda_L = \lim_{t \to 0^+} P\left[Y \le G^{(-1)}(t) \mid X \le F^{(-1)}(t) \right], \tag{16}$$

where F and G are distribution functions of continuous random variables X and Y, respectively. The power of (-1) designs the inverse of the distribution function. Therefore, the dependence index of upper (lower) tails is a limit of conditional probability of the random variable Y being higher than the quantile t of the distribution function G subject to the condition that X is higher than the quantile t of the distribution function F when t approaches 100 (0).

The second step consisted in carrying out the following algorithm.

- models AR(1)-GARCH(1,1) were estimated for the full sample (464 observations) for each series of logarithms of interest rate returns[14] $100 \cdot \ln(r_t / r_{t-1})$, t=1,…,464;
- standardised residuals $z_t = \frac{\varepsilon_t}{\sigma_t}$, t=1,…,464, were calculated;
- marginal distribution functions of these residuals were parametrically estimated;
- on the basis of these distributions functions seven copulas (Gaussian, Student's with 1, 3 and 10 degrees of freedom, Clayton's, Frank's and Gumbel's) were fitted ;
- the best copula among them was selected based on AIC and BIC criteria values.

Finally we estimated EVEaR and carried out back-g procedure for the period from September 1, 2008 to November 17, 2008 (55 observations). The algorithm of interest rate risk measurement for the purpose of calculating EVEaR was the following :

(1) Similar to the procedure on the second stage we estimated AR(1)-GARCH(1,1) models for each vector of logarithms of interest rate returns $100 \cdot \ln(r_t / r_{t-1})$, t=1,…,409;

(2) Standardised residuals in the model $z_t = \frac{\varepsilon_t}{\sigma_t}$, t=1,…,409, were computed;

(3) The marginal distribution function of these residuals was estimated by means of Student's function parametrisation ;

(4) Copula parameter was estimated (the best copula chosen on the second stage was taken here) ;

[14] As mentioned in the paper [Fantazzini (2008), p. 130], without multiplying by 100 the resulting cumulative distribution functions are not accurately estimated in the R package and elliptical copulas cannot be estimated.

(5) 1000 residuals were simulated for each interest rate for the estimated copula ;

(6) 1-day forecasts were calculated for the estimated AR(1)-GARCH(1,1) models;

(7) Residuals calculated for the copula were added to interest rate forecast computed on the basis of AR(1)-GARCH(1,1) model. This resulted in a vector of logarithms of interest rate changes ;

(8) Logarithms of interest rate changes were converted into interest rate levels r_t, t=1,…,409;

(9) Expected change in the value of equity (ΔEVE_{t+1}) was calculated by the following formula:

$$\Delta EVE_{t+1} = EVE_{t+1} - EVE_t \,, \qquad (17)$$

where $$EVE_t = \sum_i \frac{Gap_{i,t}}{(1+\frac{r_{i,t}}{366}\cdot 100)^{i/366}} \,, \qquad (18)$$

where EVE_t is the present value of a bank's equity at a time t;

$Gap_{i,t}$ is a liquidity (repricing) gap, i.e. the difference between the value of assets and liabilities at a time t for the time bucket i (for the purpose of this study we have selected five time buckets for which discount interest rates $r_{i,t}$ were applied). Gap structure for the maturity breakdown of the balance sheet is reported in table 4 below.

(10) We calculated 1% and 5% quantiles of the distribution of EVE changes (ΔEVE_{t+1}) that allows to define EVEaR value at corresponding confidence levels ;

(11) The cycle was repeated for the next time point $t+i$, i=1,…,55, when model recalculation includes an additional realisation of the previous period ;

Therefore the quality of the model when running backtesting was determined on the basis of two parameters. Firstly, we calculated the number of "breaches" in the Value-at-Risk. The drawback of this indicator is the fact that it is not sensitive to the accuracy of the forecast. Consequently, as it was shown in the papers [Pooter, Ravazzolo, Van Dijk (2007)], [Penikas (2008)] we calculated a square root of the root mean square prediction error (RMSPE) of the forecast by the following formula.

$$RMSPE_m = \sqrt{\frac{1}{\nu}\sum_{r=1}^{\nu}(\hat{y}_{T+h,m|T-r,m} - y_{T+h-r})^2}\ , \qquad (19)$$

where $RMSPE_m$ is a root mean square prediction error of the forecast for the model m;

$\hat{y}^{(T)}_{T+h,m|T-r,m}$ is a forecast value of ΔEVE_{t+1} for the model m, calculated for h steps forward (a step of one day forward was considered) when making forecast for a time $T-r$;

$y^{(T)}_{T+h-r}$ is an effective value of ΔEVE_{t+1} at a time $T+h-r$;

ν is a number of periods under consideration;

All the calculations were performed using R statistical package (version 2.8.0) that made it possible to analyse multi-dimensional copulas whereas before only the analysis of two-dimensional copulas could be performed by means of *fcopulae* package. In this paper we will comment on the possibilities of R package regarding copula modelling.

4. Data description

For measuring the interest rate exposure we use as an example of the liquidity gap[15] in accordance with expected maturities as reported in Sberbank's[16] audited financial statements as at December 31, 2007. Let us assume that the

[15] It is more reasonable to measure the interest rate risk exposure on the basis of the repricing gap, however, this information is not reported in the bank's financial statements. To this end, the authors make an assumption that the bank had a balanced structure of its floating-rate assets and liabilities.
[16] Source –Sberbank's official web site: http://www.sbrf.ru/common/img/uploaded/files/info/FS_ue2007_rus_28_04_08_cons.pdf, p. 76.

maturity structure has not changed over time. On one hand, it is a reasonable assumption if the bank's treasury sticks to the rule of maintaining the unique profile of the balance sheet's maturity structure. On the other hand, this study aims at demonstrating the techniques for measuring the interest rate risk exposure. Therefore, any bank should be able to apply the approach described here if using the repricing gap updated on a regular basis.

Thus the gap profile shown in table 3 was a starting point.

Table 3. Maturity analysis of Sberbank's balance sheet as at December 31, 2007 (mln roubles).

	Less than 1 month	1 to 6 months	6 to 12 months	1 to 3 years	More than 3 years	No maturity	TOTAL
Assets	959030	688068	1114041	1020340	980880	166449	4928808
Liabilities	484959	707538	417081	1030071	1647886	4076	4291611
Liquidity gap	**474071**	**-19470**	**696960**	**-9731**	**-667006**	**162373**	**637197**

Note : exchange rate as at December 31, 2007 : 1 USD = 24.511 RUB

The amount of liquidity gaps for assets and liabilities with no maturity was conservatively classified as "less than 1 month". Consequently, the table below shows the balance sheet profile used for measuring the interest rate risk exposure.

Table 4. Maturity profile of Sberbank's balance sheet as at December 31, 2007 (bln roubles).

Liquidity gap	**Overnight (ON)**	**1 month (M1)**	**6 months (M6)**	**1 year (Y1)**	**3 years (Y3)**
Gap (i)	636,444	-19,470	696,960	-9,731	-667,006

Note : exchange rate as at December 31, 2007 : 1 USD = 24.511 RUB

In accordance with gap maturities relevant interest rates were applied : for less than 1 year – MosPrime[17], for 1 year and more – values of zero-coupon OFZ

[17] 6-month MosPrime amounted to 6,5% for each day of the period from June 27, 2008 to July 24, 2008 (20 observations). Consequently, the estimated covariance matrix of errors for AR-GARCH model could not be calculated due to the insignificant variability of data. For this reason random numbers on [0,001;0,004] were added

yield curve[18]. The sample includes 464 observations (from January 17, 2007 to November 17, 2008). Below are descriptive statistics for interest rates that reflect stylised facts about the yield curve such as an increase in autocorrelation when the maturity extends, a decrease in interest rate volatility when debt maturity rises.

Table 5. Descriptive statistics for interest rates time series

Maturity	MosPrime Overnight	MosPrime 1 month	MosPrime 6 months	OFZ 1 year	OFZ 3 years
Mean	4,97	6,29	7,10	5,99	6,47
Standard deviation	2,19	2,55	2,49	0,80	0,81
Min	2,17	3,97	5,11	4,93	5,76
Max	22,67	20,83	21,92	10,08	9,85
Max - Min	20,50	16,86	16,81	5,15	4,09
Excess	14,75	10,00	12,86	7,24	4,70
Asymmetry	2,65	2,96	3,42	2,57	2,36
Number of observations	458	463	462	460	460
Number of gaps[19]	6	1	2	4	4
Jarque-Bera (p-value)	4646,1 (<2,2e-16)	2558,3 (< 2,2e-16)	4021,6 (< 2,2e-16)	1495,4 (< 2,2e-16)	844,7 (< 2,2e-16)
ACF (1)	0,7684	0,9512	0,9516	0,9319	0,9626
ACF (6)	0,5608	0,8104	0,7895	0,7783	0,8554
ACF (24)	0,3848	0,3292	0,3118	0,3988	0,5994

to interest rates on this interval so that when rounding to the second decimal digit these values were equal to 6,5% (source of MosPrime quotes – the Bank of Russia web site: http://www.cbr.ru/hd_base/MosPrime.asp).

[18] Source of OFZ returns: http://www.cbr.ru/GCurve/

[19] Missing observations were linearly interpolated on the basis of two closest observations available.

We would like to note that the choice of MosPrime rates from a range of interbank interest rates was driven by the fact that this rate reflects the cost of debt for the banks with sovereign credit rating which include Sberbank. An overview of existing interbank interest rates is shown in the table below.

Table 6. Comparison[20] of key Russian interbank rates

Rate	**MosPrime**	**MosIBOR**	**MIBID**	**MIBOR**	**MIACR**
Lending rate type	Reference rate for borrowing (lending)	Reference rate for borrowing (lending)	Reference rate for **borrowing**	Reference rate for **lending**	**Effective rate** on lent money
Borrowers that may get a loan at the chosen rate	Leading financial institutions	All banks	All banks	All banks	All banks
Number of banks in a poll	10	16	31		
Common banks	1.ABN AMRO, 2.WestLB Vostok, 3.VTB, 4.Gazprombank, 5.MMB (Unicredit), 6.Bank of Moscow, 7.Raiffeisenbank, 8.Sberbank, 9.Citibank				
		1.Alfa-bank, 2.Rosbank, 3.Evrofinans-Mosnarbank, 4.Trust, 5.MDM, 6.Petrokommerz			
Differing banks	1.HSBC (RR)	1. Vnesheko-nombank	1.Binbank, 2.BKF, 3. VTB24, 4.Deutsche Bank, 5.Evrotrust, 6.Zenit, 7.ING, 8.Kommerzbank (Eurasia), 9. Mezhprombank, 10. EBRD, 11.Okean, 12.NRBank, 13.Nomos, 14.Razvitie-Stolitsa, 15.Probiznesbank, 16.TransKreditBank		
Number of cut-off prices	**2** (1 on the top and 1 in the bottom)	**8** (4 on the top and 4 in the bottom)	0		
Calculation method	Average				Weighted average
Maturity from	tomorrow	today	yesterday[21]		
Published terms[22]	ON, 1W, 2W, 1M, 2M, 3M, **6M**	ON, 1W, 2W, 1M, 2M, 3M	ON, 1W, 1M, 3M, 6M, **1Y**		
Publication in Reuters system	12:31	12:30	-	-	-

[20] Source: www.cbr.ru , www.nva.ru
[21] Maturity from yesterday means that the information is published the next day after its disclosure by banks.
[22] Abbreviations for borrowing terms : ON – overnight; W – week, M – month, Y - year.

5. Results of econometric research

In our study we have decided to choose AR(1)-GARCH(1,1) model in logarithms of growth rates since, even despite insufficiently significant coefficients and the model instationarity ((α_1+β_1)>1), it approximates rather closely the distribution of residuals that is illustrated by the graphs shown below (refer to figure 2) and χ^2 test that does not allow rejecting the null hypothesis that the distribution of residuals follows the Student's *t*-distribution.

Table 7. Estimation results of the final AR-GARCH model.

Maturity	Overnight	1 month	6 months	1 year	3 years
μ (*t*-stat)	-0,0923 (-0,226)	-0,0367 (-0,722)	-0,0043 (-0,207)	0,0817 (0,829)	0,0592 (0,955)
ar(1) (*t*-stat)	0,0620 (1,373)	0,2663* (6,002)	0,0937 (1,810)	-0,2323* (-5,070)	-0,3018* (-6,485)
ω (*t*-stat)	91,6125 (1,530)	2,1834*** (2,529)	0,1457 (0,884)	0,1021 (0,874)	0,0973 (1,162)
α_1 (*t*-stat)	1,000*** (1,690)	1,000* (2,982)	1,000** (2,537)	0,0677** (2,343)	0,1448** (2,018)
β_1 (*t*-stat)	0,5134* (5,343)	0,2795* (3,837)	0,5673* (4,537)	0,9384* (26,946)	0,8600* (12,410)
Number of the degrees of freedom of *t*-distribution	2,3019* (1,402)	2,3781* (15,458)	2,2862* (18,263)	3,4318* (4,851)	3,8806* (4,783)
AIC	8,0033	3,9587	2,3026	4,7465	3,9049
BIC	8,0569	4,0124	2,3562	4,8001	3,9586
LogLikelihood	-1846,771	-910,4487	-527,0438	-1092,811	-897,992
χ^2 for residuals (p-value)	213443 (0,2398)	205572 (0,2439)	186126 (0,2546)	213443 (0,2398)	211591 (0,2408)

Note:

* significant at a 1% significance level

** significant at a 5% significance level

*** significant at a 10% significance level

Highlighted cells contain coefficients that are not statistically significant.

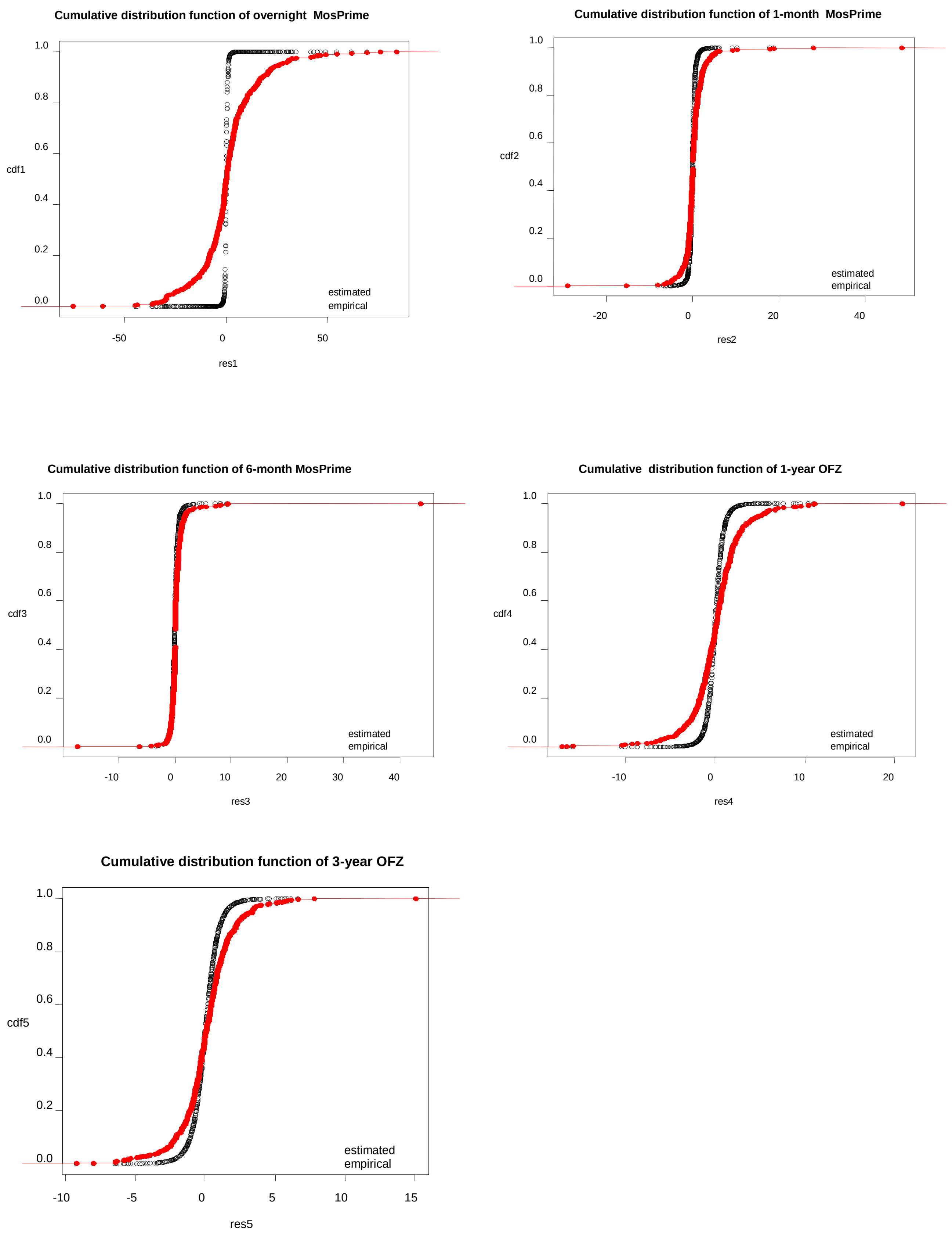
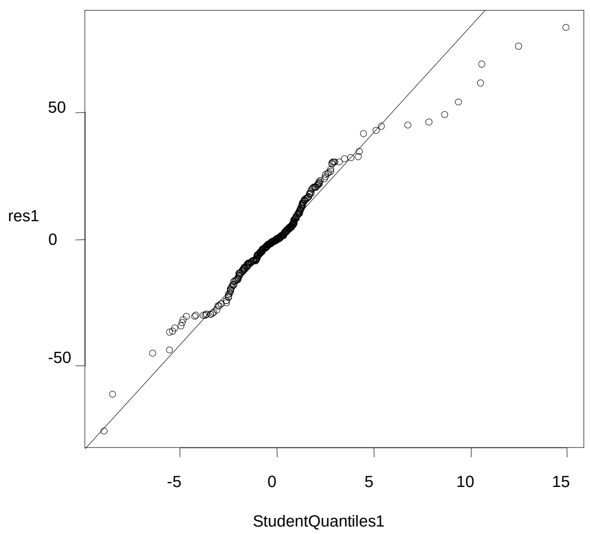


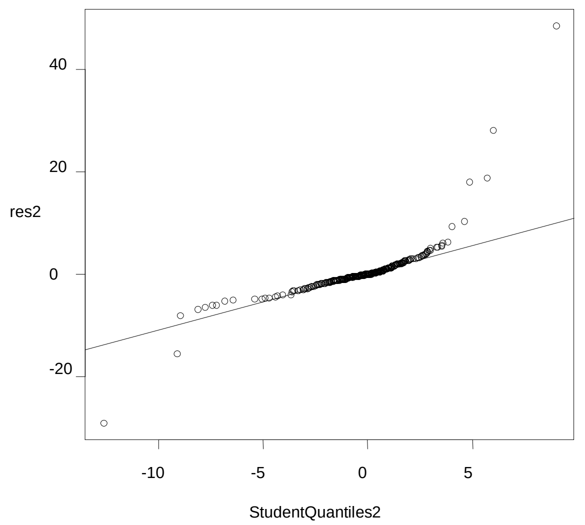
Figure 2. Empirical and estimated cumulative distribution functions of residuals.

As it appears from "quantile-quantile" graphs, the standard AR-GARCH model does not describe sufficiently well the tails of the distribution that leads to an increase in breaches of EVEaR (we will show it later). The distributions of tails that is worst approximated by *t*-distribution corresponds to the most volatile overnight rate.

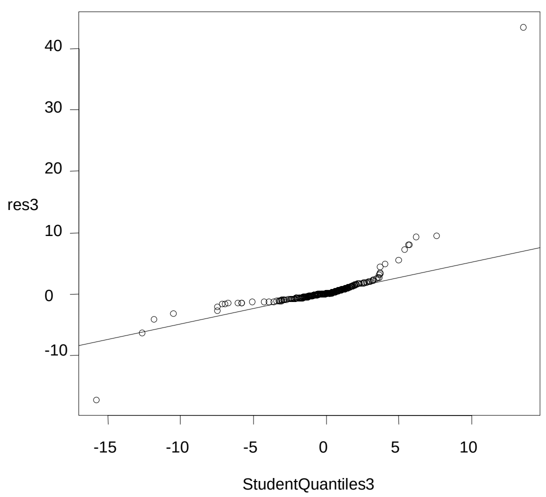
Quantile-quantile for overnight MosPrime



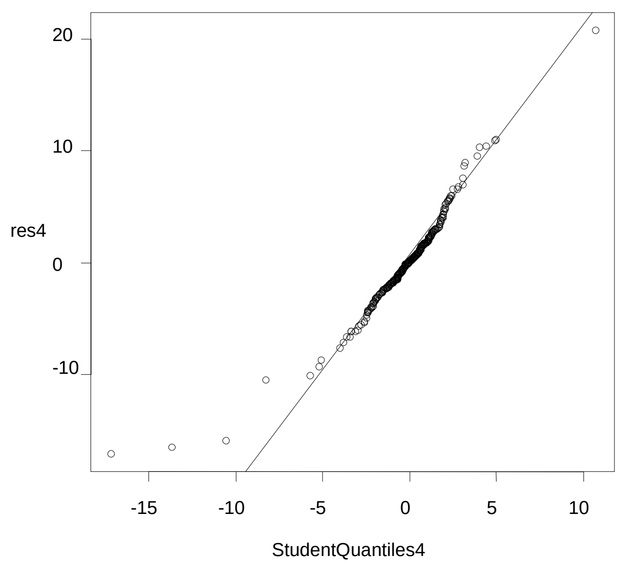
Quantile-quantile for 1-month MosPrime



Quantile-quantile for 6-month MosPrime



Quantile-quantile for 1-year OFZ



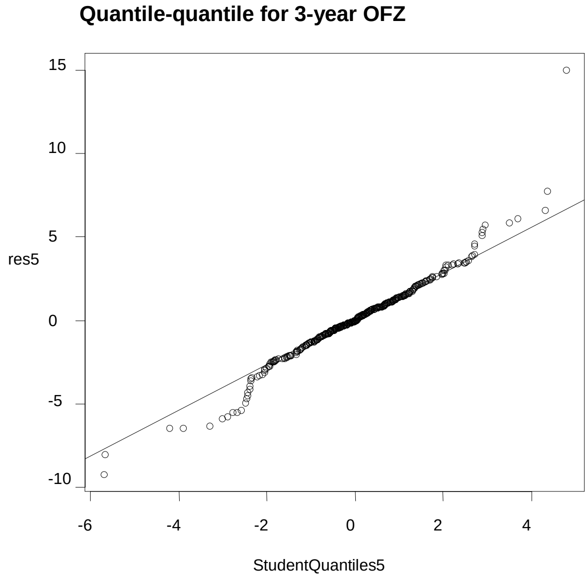


Figure 3. Graphs "Quantile-quantile" for five interest rates.

On the basis of the estimated vector of residuals we estimated[23] five-dimensional copulas (in accordance with the five time buckets under consideration). The result of this estimation is shown in the table below. It is worth noting that the parameters of all the copulas are statistically significant.

[23] All the copulas were estimated by maximum likelihood techniques (`fitCopula(…, method="ml")`), since `"mpl"` method (that is set by default as it is the least time-consuming) provides the same estimates of parameters, yet their standard deviation (`s.e.`)is approximately 1000 times higher for `"mpl"`. Estimates for `"itau"` method are different from estimates for `"irho"` by the third decimal digit. They exceed those obtained by `"ml"` techniques by approximately 10 times. In these two methods the maximum likelihood function is not calculated.

- for the normal copula the initial value of `"param"` argument is set at zero level (estimates obtained by the functions `gofCopula` and `fitCopula` are the same, so the latter was selected)

- `"param"` in the `fitCopula` function is a copula's parameter (in particular, it is a coefficient of correlation for elliptical copulas ; $\hat{\theta}$ for Clayton's copula ; $\hat{\vartheta}$ for Frank's copula ; $\hat{\delta}$ for Gumbel's copula);

- the initial value of `"param"` for t-copula, Clayton's and Gumbel's copulas was set at zero level. By means of `fitCopula` function we estimated the parameter and used it instead of the initial value. Then `fitCopula` was applied for the second time and the parameter of tail dependence was calculated for the obtained estimates.

- for t-copula the number of degrees of freedom was set to be fixed (`df.fixed = TRUE`). Otherwise when estimating t-copula by `"ml"` method there appears an error message and calculation procedure fails.

- we would like to mention the following particular feature of Frank's copula estimation in R package version 2.8.0. This copula can only be estimated in a dimension that is less than six times higher than the value of random variables.

- all options except for `method` in `fitCopula(…)` function were set by default.

The complete program code is available from the authors upon request.

Table 8. Comparison of estimation parameters for different copulas.

Copula	Copula parameter	Upper Tail Dependence Index	Low Tail Dependence Index
C_{norm}	0,1205	0	0
s.e.	(0,0067)	-	-
z-value	(18,0449)	-	-
C_t (df=1)	0,0261	0,3022	0,3022
s.e.	(0,0137)	-	-
z-value	(1,9020)	-	-
C_t (df=3)	0,0772	0,1378	0,1378
s.e.	(0,0126)	-	-
z-value	(6,1374)	-	-
C_t (df=10)	0,1184	0,0300	0,0300
s.e.	(0,0104)	-	-
z-value	(11,3960)	-	-
$C_{Clayton}$ ($\hat{\theta} \in (0;+\infty)$)	0,0894	0,0004	0
s.e.	(0,0063)	(0,0002)	-
z-value	(14,2597)	(1,8011)	-
C_{Frank} ($\hat{\vartheta} \in (-\infty;+\infty)$)	0,7958	0	0
s.e.	(0,0698)	-	-
z-value	(11,4025)	-	-
C_{Gumbel} ($\hat{\delta} \in [1;+\infty)$)	1,0333	0	0,0441
s.e.	(0,0047)	-	(0,0059)
z-value	(218,8280)	-	(7,4031)

As it can be seen from the table, an increase in degrees of freedom in Student's copula leads to a decrease in the dependence strength of tails. Meanwhile Clayton's copula is characterised by the existence of dependence of lower tails whereas Gumbel's copula – of upper tails. Gaussian copula does not capture the dependence of tails, therefore it cannot be applied for modelling extremely volatile observations during crisis periods.

Table 9. Choosing the best copula model.

Parameter of a model	C_{norm}	C_t			$C_{Clayton}$	C_{Frank}	C_{Gumbel}
		df=1	df=3	df=10			
Maximum likelihood function	212,7361	-4191,89	9,4770	0,0133	189,6838	106,3348	230,5258
AIC	-423,4722	8385,7800	-16,9540	1,9734	-377,3676	-210,6696	-459,0516
BIC	-419,3345	8389,9177	-12,8163	6,1111	-373,2299	-206,5319	-454,9139

As the values of AIC and BIC coefficients suggest, the best copula for estimating the joint distribution of the residuals of interest rate dependence model is Gumbel's copula. This result is consistent with our original intuitive assumptions. Indeed, a strong dependence of upper tails for Gumbel's copula means that the probability of a simultaneous growth in interest rates exceeds that of a simultaneous decline (that could be observed on a regular basis in autumn 2008 during the peak of the liquidity crisis in a Russian interbank loan market).

The next step of our research was to run back-testing of the interest rate risk exposure. The results of the comparison between the actual value of ΔEVE_{t+1} and the value of EVEaR calculated on the basis of AR-GARCH model under the assumption of symmetric and asymmetric distribution of tails are reported on figure 4 and in table 10.

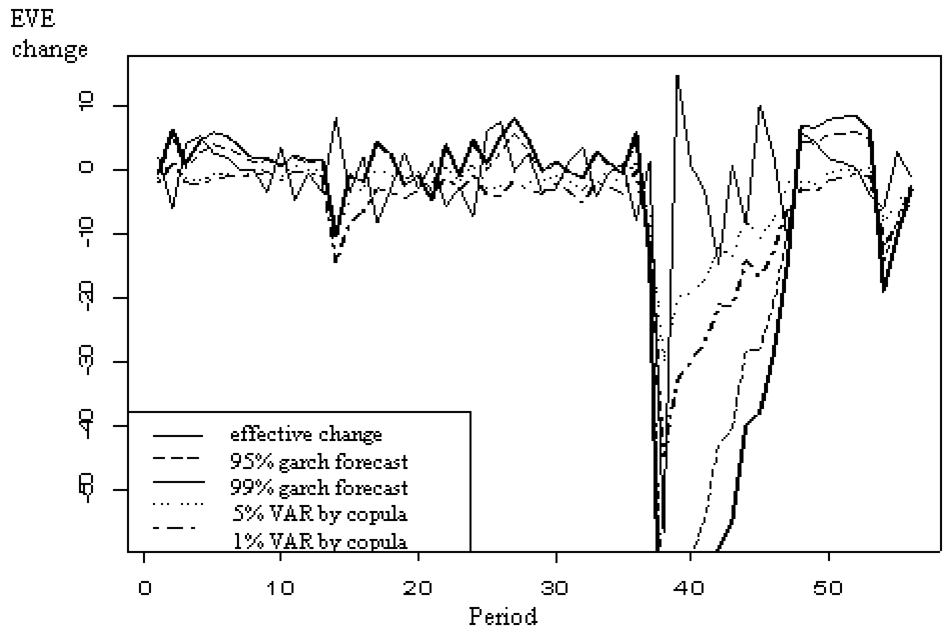


Figure 4. Effective and forecast changes in EVE (bln roubles).

Figure 4 shows a significant breach for the 38th observation. In order to understand the reason for that let us analyse the interest rate dynamics for the period under consideration (refer to figure 5). As it appears from the graph an abrupt rise in short-term interest rates on October 20, 2008 from 7% to 22% caused a drastic decline in the present value of equity. This time point is considered to be the culmination of liquidity crisis in Russian banking community when the demand for cash was caused by the necessity of making quarterly value-added tax (VAT) payments and external debt reimbursements.

From the point of view of forecasting the expected value of equity at risk, it is important to mention that the forecasts from different models are not equivalent that is confirmed by the values of root mean squared prediction error from table 10. Thus, conditional heteroscedasticity models were characterised by a long memory and a slow adjustment to interest rate changes while copula models provided more adequate results. RMSPE for GARCH amounted to 22,3% and 30,4% of Sberbank's equity (equal to 637 bln of roubles ($26 bln) as at December 31, 2007) for 95% and 99% confidence levels, respectively, while for copula models the similar values constituted 10,8% and 12,9%, respectively.

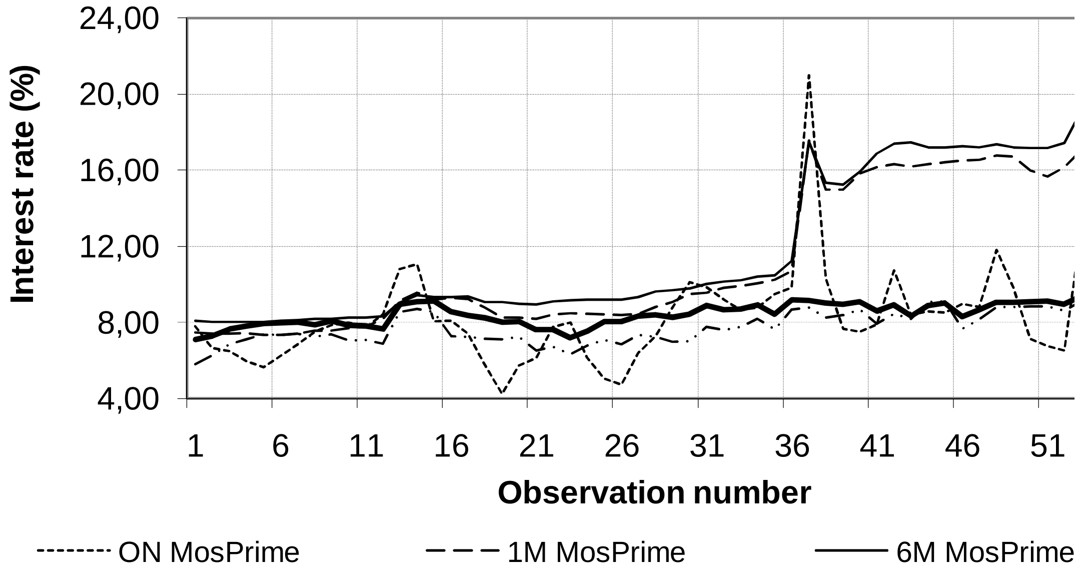


Figure 5. Interest rate dynamics from September 1, 2008 to November 17, 2008.

We would like to note that the result of the estimate by Gumbel's copula was more conservative that allowed decreasing the number of "breaches" (refer to table 10).

Table 10. Number of breaches for EVEaR[24] model.

		Confidence level EVEaR					
		95%			**99%**		
		Number of breaches	**Percentage of observations non-breaching EVEaR**	**RMSPE (bln roubles)**	**Number of breaches**	**Percentage of observations non-breaching EVEaR**	**RMSPE (bln roubles)**
AR(1) -GARCH(1,1) model		28	49%	142,11	27	51%	193,30
Copula	Gaussian	21	62%	68,91	14	75%	83,17
	Student's (1 degree of freedom)	21	62%	68,81	**13**	**76%**	85,72
	Student's (3 degrees of freedom)	**20**	**64%**	68,55	15	73%	84,18
	Student's (10 degrees of freedom)	21	62%	**68,14**	16	71%	82,27
	Clayton's	**20**	**64%**	68,37	15	73%	82,64
	Frank's	21	62%	68,50	17	69%	82,62
	Gumbel's	**20**	**64%**	68,67	15	73%	**82,24**

Note: The best characteristics of copulas are in bold.

The analysis of the number of breaches and of the RMSPE value leads to two important observations.

(1) Use of copulas increases the quality of the estimation of the value at risk for the change in the expected value of equity by 7-13% (depending on the selected confidence level);

(2) The number of breaches exceeds the selected confidence level. Even for the best copula the number of breaches comes to 36% for the 99% confidence level. This can be explained by the following reasons. First, as it was mentioned in Fantazzini's papers [Fantazzini (2009), p. 16], [Fantazzini (2008), p. 131], the best approximation of the distribution quantile is achieved by generating 100,000 observations, however due to a low speed of R package the number of simulations in this study was limited to 1,000 points. Secondly, Russian financial time series data suffers much from being

[24] Due to the fact that the residuals for the copula were obtained by random generation techniques the number of breaches can vary insignificantly (±1 breach).

rather short and possessing more missing (unregistered) values than European data (e.g. LIBOR historical quotes).

6. Main conclusions

1. We have shown that the joint distribution of Russian Interbank interest rates is asymmetric. Moreover, it is confirmed that the probability of a simultaneous increase in interest rates exceeds the probability of a simultaneous decline that is observed during crisis periods.
2. Consequently, modelling under the assumption of elliptical distributions is not reasonable. The best approximation is achieved when applying Archimedean Gumbel's copula.
3. The paper also discusses the methodology of measuring the interest rate risk exposure on the basis of the expected value of equity at risk (EVEaR) for a Russian bank;
4. We have shown that use of copulas for measuring the interest rate risk exposure for Russian Bank increases the quality of the model (in terms of the decline in the number of EVEaR breaches) by 7-13%.

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