Towards a Well-diversified Risk Measure: A DARE Approach*

Benjamin Hamidi† Patrick Kouontchou‡ Bertrand Maillet§

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- (Preliminary Version, Work-in-Progress) -

Abstract

The main objective of this paper is to provide a complete framework to aggregate different quantile and expectile models, aiming to define more diversified VaR and ES measures whilst limiting the model risk. Following Taylor (2008a and 2008b) and Gouriéroux and Jasiak (2008), we present hereafter a new strategy that provides estimations for both Value-at-Risk (VaR) and Expected Shortfall (ES). The proposed approach involves the aggregation of quantile and expectile models and the use of Asymmetric Least Squares (ALS) regression, which is the least squares analogue of quantile regression. We also introduce a new class of models for the conditional VaR and ES modelling: the Dynamic AutoRegressive Expectiles (DARE). Then, we first briefly present the main literature about VaR and ES estimations, and we explain how expectiles can be used to estimate the VaR and ES in order to introduce the DARE approach. We finally apply recent tests for comparing the main traditional approaches to the DARE approach (see for instance Candelon et al., 2008).

Keywords: Expected Shortfall, Value-at-Risk, Expectile, Risk Measures.
JEL Classification: C14, C15, C50, C61, G11.

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Towards a Well-diversified Risk Measure:
A DARE Approach

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The objective of this paper is to provide a complete framework to aggregate different quantile and expectile models for reaching more diversified VaR and ES measures limiting the model risk. Following Taylor (2008a and 2008b) and Gouriéroux and Jasiak (2008), we present hereafter a new strategy which provides estimations for both Value-at-Risk (VaR) and Expected Shortfall (ES). This approach involves the aggregation of quantile and expectile models and the use of Asymmetric Least Squares (ALS) regression, which corresponds to the least squares analogue of quantile regression. We also introduce a new class of models for the conditional VaR and ES modelling: Dynamic AutoRegressive Expectiles (DARE). Then, we first briefly present the main literature about VaR and ES estimations, and we then explain how expectiles can be used to estimate the VaR and ES in order to introduce the DARE approach. We finally present recent tests used to compare these different models (Candelier et al., 2008), also applied to the DARE Approach.

JEL Classification: C14, C15, C50, C61, G11.

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INTRODUCTION

Value-at-risk (VaR) measures the potential loss of a given portfolio over a specified holding period at a specified confidence level, which is commonly fixed at 1% or 5%. The Basel Committee on Banking Supervision (1996) at the Bank for International Settlements imposes on financial institutions, such as banks and investment firms, to get required funds based on VaR estimates. Although VaR became the standard measure of market risk, it has been criticized for reporting only a quantile, and thus disregarding all the outcomes beyond the threshold. In addition, VaR is not a subadditive risk measure. This property concerns the main idea which is the aggregated risk of a portfolio should not be greater than the individual risk of its constituent parts (see Artzner et al., 1999; Acerbi and Tasche, 2002). The Expected Shortfall (ES) is a risk measure that overcomes these weaknesses. Indeed, the ES is defined as the conditional expectation of the return given that it falls below the VaR (see Yamai and Yoshina, 2002).

The objective of this paper is to provide a complete framework to aggregate different quantile and expectile models, to provide more diversified VaR and ES measures, limiting the model risk in the same way it does for the volatility (see Visser, 2008).

A recent development in the VaR literature concerns the Conditional Autoregressive Value-at-Risk (CAViaR) class of models (see Engle and Manganelli, 2004). This approach to the VaR estimation has a strong appeal insofar as it provides an extensive modelling framework and does not rely (too much) on strong distributional assumptions. However, the focus is merely on VaR estimation, and it is not obvious to estimate the corresponding ES.

Following Taylor (2008a and 2008b) and Gouriéroux and Jasiak (2008), we present in this paper a new modelling approach that defines estimations for both VaR and ES. The approach involves the use of Asymmetric Least Squares (ALS) regression. The solution of an ALS regression is known as an expectile. This name was given by Newey and Powell [1987] who note that the ALS solution is determined by the properties of the expectation of exceedances beyond the solution. We use in the following this result to estimate the ES. It has also been shown that there exists a one-to-one mapping from expectiles to quantiles. Indeed, Efron [1991] proposes the $\alpha$-quantile to be estimated by the expectile for which the proportion of in-sample observations lying below the expectile is equal to $\alpha$. This idea can also be used to estimate the VaR from expectiles as shown later on.

This main aim of the article is to present an extension of the conditional VaR models provided by the CAViaR approach, as we propose a new class of models for the conditional VaR and ES modelling: the Dynamic AutoRegressive Expectiles (DARE in short). In the first section, we review the literature about VaR and ES measures and estimation models. In the second section, we describe how expectiles can be used to estimate VaR and ES, then introduce the DARE approach. We finally present in the third section the empirical results regarding the accuracy of the proposed method, based on the different usual tests of models of VaR and ES. The last section concludes.
MARKET RISK MODELS

We introduce hereafter the extreme market risk measures, that are the Value-at-Risk (VaR) and the Expected Shortfall (ES), focusing on the way they allow us to disentangle the behavior of tails.

Risk management has become in these past few years a central object of interest for researchers, market practitioners and regulators. The Value-at-Risk (VaR) is well known as being one of the benchmark measure for market risk.

The VaR is defined as the loss a portfolio may suffer at a given confidence level over a fixed holding period such as:

\[ \text{Prob}[r_t < -VaR_t(\alpha)] = 1 - \alpha, \]  

where \( r_t \) is the random variable of asset returns, and \( \alpha \) is the confidence level for the VaR computation.

The distribution of \( r_t \) can be set here as an empirical non-continuous distribution, or as a theoretical specified continuous distribution.

The Value-at-Risk of an empirical distribution can also be viewed as the \( \alpha \)-quantile. Although the VaR can be mentioned as a reference and is imposed by regulators, criticisms have been recently formulated against its generalized use. For instance, the VaR does not enable us to answer the question: “what is the magnitude of the loss when the VaR limit is exceeded?”.

Another issue regarding the VaR, pointed out by the article of Artzner et al. [1999], concerns the “non-coherence” property of this risk measure, it fails to respect the subadditivity property, i.e. the VaR of two assets in a portfolio can be greater than these two aggregated individual VaRs. Prause (1999) also argues that, to avoid bankruptcy, we should forecast the distribution of the maximum expected loss. From this point of view, regulators should use other risk measures than the VaR in order to get a better characterization of extreme events, especially for nonlinear portfolio returns. Therefore, another risk measure has appeared in the literature, the Expected Shortfall, that is the value of the expected loss at a given confidence level. More formally, the Expected Shortfall (ES) can be written as such (with previous notations):

\[ ES_{\alpha,t}(r_t) = -E[r_t | r_t \leq VaR_t(\alpha)], \]  

If we consider that the distribution of \( r_t \) is known, we have:

\[ ES_{\alpha,t}(r_t) = \frac{1}{1 - \alpha} \int_{-\infty}^{-VaR_t(\alpha)} xf(x)dx, \]  

where \( f(\cdot) \) is the probability density function of \( r_t \).

Contrary to VaR, this new measure satisfies the subadditivity property mentioned above, but leads to a new question: “How bad is bad?”.

Several approaches serve for estimating these market risk measures. The choice will depend on the kind of portfolio, computational resources available and time constraints. The three main approaches can be classified into three
categories: parametric, non-parametric and semi-parametric methods.

The parametric approach assumes that the returns have a specific probability distribution, as for example for the Normal or Student law. The parameters of the distribution, like volatility, are estimated and the risk measure is then calculated, based on the estimated distribution. Consequently, the risk measure mainly depends on the parameters used. The Normal and Student distribution are the most used law with this approach. The RiskMetrics model is the most used market risk model, assuming that asset returns follow a centered Normal distribution with a volatility estimated by the Exponential Weighted Moving Average (EWMA) method, obtained by using historical data. Many other volatility estimation methods, belonging to the GARCH family, have been already tested to replace the EWMA. This approach is called the Conditional Normal method insofar as we use a Normal distribution with the variance, calculated by conditional methods. Therefore, the Normal distribution has thinner tails than the empirical distributions which is the reason why other distributions have also been tested (Student).

The most widely used non-parametric method to estimate a predetermined quantile is based on the so-called historical simulations. The latter only requires mild distributional assumptions and implies the estimation of the VaR as the quantile of the empirical distribution of historical returns from a moving window of the most recent period. The main problem is the way to get the width of this window: few observations will lead to an important sampling error, whereas too many observations will slow down estimates to react to changes in the true distribution of financial returns. Other methods allocate to the sample of returns exponentially decreasing weights (which sum to one). The returns are then ordered in ascending order from the lowest return and the weights are aggregated to reach the given confidence rate; the conditional quantile estimate is set as the return that corresponds to the final weight used in the previous summation. The forecast is then built from an Exponentially Weighted Average of past observations. If the distribution of returns is moving quickly over time, a relatively fast exponential decay is needed to ensure a reactive adaptation. These exponential smoothing methods are simple and common approaches in practise.

Semi-parametric VaR approaches are based on Extreme Value Theory and Quantile Regression method. This estimation method does not need any distributional assumptions. The Conditional AutoRegressive VaR (CAViaR) introduced by Engle and Manganelli [2004] is one of them. They have directly defined the dynamics of risk with autoregression, involving the lagged Value-at-Risk (VaR) and value of an endogenous variable called CAViaR.

THE DARE APPROACH

The main goal of this article is to propose a way to aggregate well specified expectiles and quantile models in order to reach a good estimation of quantiles based on risk measures such as VaR and ES. We focus on these two main points to answer this issue. On one hand, we have to provide a unified framework which enables us to aggregate quantile estimation methods, involving the introduction hereafter of the DARE approach. On the other hand, we need to evaluate whether quantile based on risk measures are well specified. Our evaluation criteria are the VaR and ES tests developed in
the literature (see the technical appendix).

This section briefly reminds the definition of expectiles, and their uses to estimate ES and VaR with Conditional AutoRegressive Expectile (CARE) class of models. Then, we introduce our DARE methodology to aggregate CARE model with a unified quantile estimation method. We detail the tests used to evaluate these methods in the appendix.

VaR and Expected Shortfall can be estimated with Expectiles calculated by the minimization of:

$$\mu^*_{\tau,t} = \text{ArgMin}_{\mu_{\tau,t} \in \mathbb{R}} \left\{ E \left[ \left| \tau - I_{\{r_t < \mu_{\tau,t}\}} \right| (r_t - \mu_{\tau,t})^2 \right] \right\}, \quad (4)$$

where the population $\tau$-expectile of $r_t$ is the parameter $\mu_{\tau,t}$ and $\left| \cdot \right|$ is the absolute value.

Parameters of a conditional model for expectiles can be estimated by using the Asymmetric Least Square (ALS) regression, which is the least square analogue for quantile regression:

$$\beta^* = \text{ArgMin}_{\beta \in \mathbb{R}^n} \sum_{t=1}^{T} \left\{ \left| \tau - I_{\{r_t < \hat{\mu}_{\tau,t}(r_{t-1};\beta)\}} \right| \times \left[ r_t - \hat{\mu}_{\tau,t} (r_{t-1};\beta) \right]^2 \right\}. \quad (5)$$

where $\hat{\mu}_{\tau,t}$ is the estimation of $\mu_{\tau,t}$.

We can use expectiles as quantile estimators (and VaR) given that there is a corresponding $\alpha$-quantile for each $\tau$-expectile (see Efron, 1991; Jones, 1994; Abdous and Remillard, 1995; Yao and Tong, 1996). Taylor [2008a and 2008b] explained the link between the Expectile and Expected Shortfall, that leads to:

$$ES_{\alpha,t}(r_t;\beta) = \left[ 1 + \tau (1 - 2\tau)^{-1} \alpha^{-1} \right] \hat{\mu}_{\tau,t} \left( r_t;\beta \right), \quad (6)$$

where $\hat{\beta}$ is the estimation of $\beta$.

Thus, the conditional Expected Shortfall for a given value of $\alpha$ is proportional to the conditional $\gamma$-quantile model, which is estimated by the $\tau$-expectile.

Moreover, the quantile estimations can be linearly combined through the DAQ model proposed by Gouriéroux and Jasiak [2008]. This class of dynamic quantile models is defined by:

$$Q_{\alpha,t} (\beta) = \sum_{k=1}^{K} a_k (r_{t-1}, y_{t-1}, \beta_k) \times Q_{k,\alpha,t} (\beta_k) + a_0 (r_{t-1}, y_{t-1}, \beta_0), \quad (7)$$

where $Q_{k,\alpha,t}(\cdot)$ are path-independent quantile functions and $a_k(\cdot)$are non-negative functions of the past returns and exogenous variables.

Thus, DAQ model can use different quantile functions to model a given one. We can also combine these functions into a multi-quantile method to increase the accuracy of the conditional model. Actually, every quantile function can be extended to define a simple class of parametric dynamic quantile models.

The Expected Shortfall can therefore be expressed as a combination of quantile whose associated probabilities are defined through the estimation of equation (5):

$$ES_{\alpha,t} = W_{\alpha,t} \cdot Q_{\gamma,t},$$

$$\left[ 1 \times 1 \right] \quad \left[ 1 \times n \right] \quad \left[ n \times 1 \right]$$

$$8$$
with \( i = [1, 2, ..., n] \):

\[
\begin{align*}
\{ \mathbf{W}_{\alpha,t} &= \begin{bmatrix} w_1 \times \beta_{\alpha,1} & w_2 \times \beta_{\alpha,2} & \ldots & w_n \times \beta_{\alpha,n} \end{bmatrix}, \\
\mathbf{Q}_{\gamma,t} &= \begin{bmatrix} Q_{\gamma_1,t}^1 & Q_{\gamma_2,t}^2 & \ldots & Q_{\gamma_n,t}^n \end{bmatrix} \}
\end{align*}
\]

and:

\[
\begin{align*}
\overline{ES}_{\alpha,t} &= \beta_i Q_{\gamma_i,t}^i (\mu_{\tau_i,t}, \beta_i) \\
\beta_i &= \left[ 1 + \tau_i (1 - 2\tau_i)^{-1} \alpha^{-1} \right] \\
\sum_{i=1}^n w_i &= 1
\end{align*}
\]

where \( Q_{\gamma_i,t}^i (.) \) are several quantile functions associated with probability \( \gamma_i \).

The estimation of the \( \alpha \)-Expected Shortfall can be found by aggregating linearly several quantile functions and estimating the right correspondence between the probability \( \alpha \) associated to the Expected Shortfall and the probability \( \gamma_i \) associated to each quantile functions (thanks to the \( \tau_i \)-expectile defined by the equation 4).

Actually, given \( n \) coherent risk measures, any convex linear combination is another coherent risk measure, thus the DARE approach provides also a coherent risk measure contained in the generated convex hull. This new space of coherent measures allows us to define a coherent spectral measure of risk.

**EXPERIMENTS AND RESULTS**

For investigating the effectiveness of the DARE approach for VaR and Expected Shortfall computations, we apply the method to the French Stock Market data and compare it to established methods. We use daily prices from 9th July 1987 until 18th March 2009 of the CAC40 Index. This period delivers 5,659 daily returns. We use a moving window of four years (1,044 daily returns) to re-estimate dynamically parameters for the various methods. Forecasted VaR and ES were computed for each method for the final 4,615 days (about 18 years). This comparison considers daily estimation of the 95% and 99% conditional VaR and ES. These quantiles were chosen because they are commonly considered and controlled by financial firms and regulators.

Although there are many VaR estimation methods, we restricted our comparison to commonly used benchmark methods.

We first consider the most widely used non-parametric method: the historical simulation. It requires no distributional assumptions and estimates the VaR as the quantile of the empirical distribution of historical returns from a moving window.

Then parametric approaches, which are estimated, assume a particular shape of return distribution as Gaussian or \( t \)-Student distribution. We respectively consider the Normal, the RiskMetrics, the GARCH(1,1) VaR and ES. All these parametric methods also assume a Gaussian distribution by using various conditional volatility forecasts: respectively the empirical volatility, the RiskMetrics correlation method based on an Exponential Moving Average estimation, GARCH(1,1) model (the use of the (1,1) specification was based on the general
popularity of this order for GARCH models). We also estimate VaR and ES with the Student’s-t distribution optimizing parameters by using maximum likelihood.

CAViaR Models are presented as benchmark methods. They are estimated by using the Engle and Manganelli’s procedure [2004]. We finally compute the DARE approach by aggregating CARE Models based on CAViaR models. As mentioned in the previous section, the DARE approach can be used to estimate the VaR and ES. The relation between VaR and ES in the DARE approach is defined by the equation (6). To be sure to have a good estimation of ES using the DARE approach we have first to test VaR based on it. Table 1 presents the main tests of VaR presented in the literature (see appendix 1 for a detailed description of these tests).

The Exception Frequency test (Kupiec, 1995) is based on the Hit ratio. Table 1 shows that according to this simple test, benchmark methods as the Historical, Normal, RiskMetrics, or CAViaR Asymmetric Slope are not appropriated. We remark that combining naively every method (Mean of VaR), this criteria is respected. Aggregating CaViaR methods thanks to CARE models in the DARE approach, allows also to respect this test (the hypothesis of a good VaR can be anymore rejected).

No benchmark method can be considered as a “good” measure for every test even if some measures are definitively not appropriated. Thus, according to these tests on this data sample, the worst methods are the Historical and Normal VaR and the best measures seem to be the DARE approach for VaR, CAViaR Symmetric Absolute Value, CAViaR Indirect GARCH(1,1) and the naive aggregation (Mean of VaR).

The aggregation of VaR approaches seem to be more robust than the classical VaR computations. Thus, mixing these different benchmark methods naively does not allow to respect every criteria, but test results are more stable to the variation of the probability level associated to the VaR. Using the DARE approach allows us to respect most of these tests.

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Source: DataStream; daily data from July 9th 1987 until March 18th 2009 of the CAC40 Index; computations by the authors. This period contains 5,659 daily returns. We use a moving window of four years (1,044 daily returns) to dynamically re-estimate parameters for the various methods. Forecasted VaR were computed for each method for the final 4,615 days (about 18 years). This table shows p-values (expressed in percentage) of different tests statistics for good VaR model of the alternative VaR models under the 5% and 1% significance levels. (1) Exception Frequency is based on the indicator variable of the good VaR forecast; (2) Independence test (Christoffersen, 1998) detects the serial independence in the VaR model forecast; (3) Unconditional Coverage test (Kupiec, 1995) estimates is an simply counts exceptions over the entire period and Conditional Coverage test (Christoffersen, 1998) is based on the idea that if the VaR estimates have correct conditional coverage, the exception variable must exhibit both correct unconditional coverage and serial independence; (4) Dynamic Quantile test (Engle and Manganelli, 2004) detects a good model for VaR estimation with the ideas that a good model should produce a sequence of unbiased and uncorrelated exception indicator variable; (5) Distributional Forecast Test (Berkowitz, 2001) is based on the fact that the VaR models are characterized by their distribution forecasts of portfolio returns; (6) Exception Magnitudes (Berkowitz, 2001) is based on the idea that the magnitudes of exceptions should be of primary interest to the various users of VaR models; (7) to (9) Generalized Method of Moment Duration-based tests (Candelon et al., 2008) turn out that VaR forecast tests can be expressed as simple moment conditions.
CONCLUSION

In this paper, we present a new modelling approach for Value-at-Risk and Expected Shortfall, called Dynamic AutoRegressive Expectiles (DARE). This approach involves the use of the Asymmetric Least Squares (ALS) regression, which is the least squares analogue of quantile regression. We show that the DARE methodology advantageously reveals an underlying structure in data that can be used for risk measure computations. We also show, with different VaR and ES specifications (Normal, Historical, RiskMetrics, Student, GARCH and CAViaR) and various backtests (Conditional and Unconditional, Distributional Forecast, Duration-based tests), that the DARE model is well adapted to the financial data. Indeed, the aggregation of VaR approaches is more robust than the classical VaR and ES computations. Finally, based on our first results, the DARE approach, aggregating of several VaR measures, could have some interest in further financial applications, such as stress test assessment or asset pricing.

As a forthcoming extension of this paper, we will add some robust resampling technique tests (Escanciano and Olmo, 2009) and other specific tests specifically dedicated to Expected Shortfall testing. For instance, a recent paper by Hurlin and Tokpavi (2007) explicitly considers the Expected Shortfall in a multi-quantile test framework.

REFERENCES


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Appendix I: Quantile and ES Models

Non Parametric Models

For estimating a predetermined quantile, the most widely used non-parametric method is based on so-called historical simulations. These do not require any specific distributional assumptions and lead to estimate the VaR as the quantile of the empirical distribution of historical returns from a moving window of the most recent period. The essential problem is to determine the width of this window: including too few observations will lead to large sampling error, while using too many observations will result in estimates that are slow to react to changes in the true distribution of financial returns. Other methods involve allocating to the sample of returns exponentially decreasing weights (which sum to one). The returns are then ordered in ascending order and starting at the lowest return the weights are summed until the given confidence rate is reached; the conditional quantile estimate is set as the return that corresponds to the final weight used in the previous summation. The forecast is then constructed from an exponentially weighted average of past observations. If the distribution of returns is changing relatively quickly over time, a relatively fast exponential decay is needed to ensure coherent adapting. These exponential smoothing methods are simple and popular approaches.

Parametric Models

Parametric approaches of quantile estimation involve a parameterization of the stock prices behavior. Conditional quantiles are estimated using a conditional volatility forecast with an assumption for the shape of the distribution, such as a Student-t. A GARCH model can be used for example to forecast the volatility (see Poon and Granger, 2003), though GARCH misspecification issues are well known.

Semi-parametric Models

Semi-parametric VaR approaches are based on Extreme Value Theory and Quantile Regression methods. Quantile regression estimation methods do not need any distributional assumptions. Conditional AutoRegressive VaR (CAViaR) introduced by Engle and Manganelli (2004) is one of them. They have defined directly the dynamics of risk by means of an auto regression involving the lagged-Value at Risk (VaR) and the lagged value of endogenous variable called CAViaR. They present four CAViaR specifications: model with a symmetric absolute value, an asymmetric slope, indirect GARCH(1,1) and an adaptive form, denoted respectively: $C_{SAV_t}(r_t; \beta)$, $C_{AS_t}(r_t; \beta)$, $C_{IG_t}(r_t; \beta)$, $C_{At}(r_t; \beta)$ where:

\[
\begin{align*}
C_{SAV_t}(r_t; \beta) &= \beta_1 + \beta_2 \times C_{SAV_{t-1}}(r_{t-1}; \beta) + \beta_3 \times |r_{t-1}| \\
C_{AS_t}(r_t; \beta) &= \beta_1 + \beta_2 \times C_{AS_{t-1}}(r_{t-1}; \beta) + \beta_3 \times \max(0, r_{t-1}) + \beta_4 \times [- \min(0, r_{t-1})] \\
C_{IG_t}(r_t; \beta) &= \beta_1 + \beta_2 \times [C_{IG_{t-1}}(r_{t-1}; \beta)]^2 + \beta_3 \times r_{t-1}^2 \\
C_{At}(r_t; \beta) &= C_{At-1}(r_{t-1}; \beta) - 0.01 + \beta_1 [1 + \exp\{0.5 \times [r_{t-1} - C_{At-1}(r_{t-1}; \beta)]\}]^{-1}
\end{align*}
\]
and where \( \beta_i \) are parameters to estimate and \( r_t \) is the risky asset return at time \( t \).

Discussing the general CAViaR model without the autoregressive component, the conditional quantile function is well defined if parameters can be considered as quantile functions too. In fact, CAViaR models weight different baseline quantile functions at each date and can be therefore considered as quantile functions. Adding a non-negative autoregressive component of VaR, the CAViaR conditional quantile function becomes a linear combination of quantile functions weighted by non-negative coefficients. Thus CAViaR model satisfies the properties of a quantile function, even if the indirect GARCH(1,1) or adaptive specifications do not satisfy the monotonicity property. CAViaR model parameters are estimated using the quantile regression minimization (denotes QR Sum) presented by Koenker and Bassett (1978):

\[
\min_{\beta} \sum_t \{ [y_t - Q_t(\alpha)] [\alpha - I_{\{y_t < Q_t(\alpha)\}}] \}
\]  

(10)

with \( Q_t(\alpha) = x_t \beta \) where \( x_t \) is a vector of regressors, \( \beta \) is a vector of parameters and \( I_{\{\cdot\}} \) is an Heavyside function.

When the quantile model is linear, this minimization can be formulated as a linear program for which the dual problem is conveniently solved. Koenker and Bassett (1978) show that the resulting quantile estimator, \( \hat{Q}_t(\alpha) \), essentially partitions the \( y_t \) observations so that the proportion less than the corresponding quantile estimate is \( \alpha \).

The procedure proposed by Engle and Manganelli (2004) to estimate their CAViaR models is to generate vectors of parameters from a uniform random number generator between zero and one, or between minus one and zero (depending on the appropriate sign of the parameters). For each of the vectors then evaluated the QR Sum. The ten vectors that produced the lowest values for the function are then used as initial values in a quasi Newton algorithm. The QR sum is then calculated for each of the ten resulting vectors, and the vector producing the lowest value of the QR Sum is to be chosen as the final parameter vector.

Gouriéroux and Jasiak (2006) have introduced dynamic quantile models (DAQ). This class of dynamic quantile models is defined by:

\[
Q_t(\alpha, \beta) = \sum_{k=1}^{K} a_k(y_{t-1}, \beta_k) \times Q_k(\alpha, \beta_k) + a_0(y_{t-1}, \beta_0)
\]  

(11)

where \( Q_k(\alpha, \beta_k) \) are path-independent quantile functions and \( a_k(y_{t-1}, \beta_k) \) are non-negative functions of the past returns and other exogenous variables.

A linear dynamic quantile model is linear in the parameters, then:

\[
Q_t(\alpha, \beta) = \sum_{k=1}^{p} \beta_k Q_{k,t}(\alpha)
\]  

(12)

Thus DAQ model can use different quantile functions to model a given quantile. To increase the accuracy of the model, we can also combine different quantile functions, in a multi-quantile method. Actually, previous quantile functions can be extended to define a simple class of parametric dynamic quantile models.

But VaR is also criticized for only reporting a quantile and thus disregarding outcomes beyond the quantile. In addition VaR is not a subadditive risk measure and can therefore dissuade from diversification. This property concerns the idea that the total risk on a portfolio should not be greater than the sum of the risks of the constituent parts of the portfolio.
Appendix II: Backtesting Quantile Models

The VaR Evaluation tests used in this paper cover the Unconditional and Conditional, the Dynamic Quantile, the Distributional Forecast and the Exception Magnitude tests.

For the latter, the exception indicator variable associated to the ex-post observation of $\alpha$-VaR violation at the time $t$, $I_t(\alpha)$, is defined such by:

$$I_t(\alpha) = \begin{cases} 1 & \text{if } r_t < VaR_{t-1}(\alpha) \\ 0 & \text{otherwise} \end{cases}$$

where $r_t$ the return at the time $t$ and $VaR_{t-1}(\alpha)$ the one-period VaR computed for the $\alpha$-quantile at the time $t-1$.

The problem of VaR validity can be therefor tests by the knowing whether the violation sequence $I_t(\alpha)$, $t = [1, \ldots, T]$, for a given $\alpha$-quantile.

II.1 Unconditional and Conditional Exceptions Tests

The most frequently backtests of VaR models, based on the exception indicator, are the Unconditional and Conditional Tests. These VaR tests cover Unconditional Coverage Test (Kupiec, 1995), Independence Test and Conditional Coverage Test (Christoffersen, 1998). The conditional coverage test combines the Unconditional Coverage Test and the Independence Test.

An Unconditional Coverage Test (Kupiec, 1995)

The Unconditional coverage test of VaR estimates is a count of exceptions over the entire period. Kupiec (1995) shows that if the VaR model is well specified then the exceptions occurred, can be modelled as independent draws from a binomial distribution with a probability of occurrence equal to $\alpha$-percent. The Likelihood Ratio statistic is:

$$LR_{uc} = 2\{\log[\hat{\alpha}N_T(1 - \hat{\alpha}^{T-N_t})] - \log[\alpha^{N_T}(1 - \alpha^{T-N_t})]\}$$

where $T$ the number of observations, $N_T = TE[I_t(\alpha)]$ the number of exception and $\hat{\alpha} = N_T/T$ the unconditional coverage.

The Likelihood Ratio $LR_{uc}$ has an asymptotic $\chi^2(1)$ distribution with the hypothesis $\hat{\alpha} = \alpha$.

An Independence Test (Christoffersen, 1998)

This test detects the serial independence in the VaR forecast model. VaR violations observed at two different dates for the same coverage rate must be distributed independently. Christoffersen (1998) suggests that the process of $I_t(\alpha)$ violations is modeled with a Markov chain with the following transition probabilities matrix:

$$\Pi = \begin{pmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{pmatrix},$$

where $\pi_{ij} = Prob[I_t(\alpha) = j | I_{t-1}(\alpha) = i]$. 

---

*In this section, we only gives a summary of the tests. For more details about these tests, see Christoffersen (1998); Christoffersen and Pelletier (2003) or Ferreira and Lopez (2005).*
The Likelihood Ratio for the test is:

\[ LR_{ind} = 2\log L(\pi_{01}, \pi_{11}) - \log L(\pi, \pi) , \]  

which has an asymptotic \( \chi^2(1) \) distribution (Christoffersen, 1998). \( L(\pi_{01}, \pi_{11}) \) is the likelihood under the hypothesis of the first-order Markov dependence:

\[ L(\pi_{01}, \pi_{11}) = (1 - \pi_{01})^{T_{00}} \pi_{01}^{T_{01}} (1 - \pi_{11})^{T_{10}} \pi_{11}^{T_{11}} , \]

with \( T_{ij} \) denotes the number of observations in the state \( j \) after having been in state \( i \) in the previous period, and with \( \pi_{01} = T_{01}/(T_{00} + T_{01}) \) and \( \pi_{11} = T_{11}/(T_{10} + T_{11}) \). On the contrary, \( L(\pi, \pi) \) is the likelihood under the hypothesis of independence (\( \pi_{01} = \pi_{11} = \pi \)):

\[ L(\pi, \pi) = (1 - \pi)^{T_{00} + T_{10}} \pi^{T_{01} + T_{11}} , \]

with \( \pi = (T_{01} + T_{11})/T \).

**A Conditional Coverage Test** (Christoffersen, 1998)

The Conditional test used hereafter is a test of conditional coverage proposed by Christoffersen (1998). In this case, if the VaR estimates have a correct conditional coverage, the exception variable must exhibit both correct unconditional coverage and serial independence. The Conditional Coverage test \( LR_{cc} \) is therefore a joint test of these properties and the relevant statistic is, with the previous notations:

\[ LR_{cc} = LR_{uc} + LR_{ind} , \]

which is asymptotically i.i.d. distributed as a \( \chi^2(2) \).

**II.2 A Distributional Forecast Test** (Berkowitz, 2001)

This test is based on the fact that VaR models are generally characterized by their distribution forecasts of portfolio returns; thus, the evaluations should be based directly on these forecasts. The interest of these methods is the observed quantile \( \tau_t \), which is the quantile under the distribution forecast \( f_t \) for which the observed return \( r_t \) actually falls. If the underlying VaR model is accurate, then its \( \tau_t \) series should be independent and uniformly distributed over the unit interval. The observed quantile \( X_{t+1} \) is defined as:

\[ \tau_t = F_{t-1}(r_t) , \]

with \( VaR_{t-1}(\alpha) = F_{t-1}^{-1}(\alpha) \).

For testing the two properties of observed quantiles series (independence and uniform distribution), Diebold et al. (1998) propose the use of CUMSUM statistic, Christoffersen and Pelletier (2003) a Duration-based tests, and Berkowitz (2001) propose a Likelihood Ratio Test used in this paper.

In order to simultaneously apply these two properties, Berkowitz (2001) also proposes the transformation \( z_t = \Phi^{-1}(\tau_t) \); which corresponds to the inverse of the standard normal cumulative distribution function of \( \tau_t \). Under the null hypothesis, the \( LR \) statistic is:

\[ LR_{dist} = 2[L(\mu, \rho, \sigma^2) - L(0, 0, 1)] , \]
which is an asymptotically distributed $\chi^2(3)$, with:

$$ L(\mu, \rho, \sigma^2) = \sum_{t=1}^{T} \{-1/2 \log[(2\pi\sigma^2)(1-\rho^2)^{-1}] - L_z \} $$

$$ -1/2(T-1)\log(2\pi\sigma^2) - \sum_{t=1}^{T-1} (2\sigma^2)^{-1}(z_{t+1} - \mu - \rho z_t)^2, $$

with $L_z = \{[z_{t+1} - \mu(1-\rho)^{-1}]^2[2\sigma^2/(1-\rho^2)]^{-1}\}$ and where $(\mu, \rho)$ are respectively the conditional mean and AR(1) coefficient corresponding to the $z_{t+1}$ series; i.e. $z_{t+1} = \rho(z_t - \mu) + \epsilon_{t+1}$, with $\epsilon_{t+1}$ the normal random variable with zero mean and variance $\sigma^2$.

II.3 An Exception Magnitude Test (Berkowitz, 2001)

The main idea of this test is the magnitude of exceptions should be of primary interest to the various users of VaR models (see Berkowitz, 2001; Ferreira and Lopez, 2005). In the same way as the $LR_{dist}$ test, Berkowitz (2001) transforms the empirical series into standard normal $z_t$ series. The $z_t$ values are then compared to the normal random variables with the desired coverage level of the VaR estimates. If the VaR model generating the empirical quantiles is correct, the $\gamma_t$ series, defined by (21), should be identically distributed and $(\mu, \sigma)\footnote{\mu$ and $\sigma$ are respectively the unconditional mean and standard deviation of the $z_t$ series.}$ should be equal to $(0,1)$.

$$ \gamma_{t+1} = \begin{cases} z_t & \text{if } z_t < \Phi^{-1}(\alpha) \\ 0 & \text{otherwise} \end{cases}, \quad (21) $$

where $\Phi$ is the standard normal cumulative distribution function. Finally, the corresponding test statistic is:

$$ LR_{mag} = 2[\text{Mag}(\mu, \sigma) - L(0,1)], $$

which is asymptotically distributed $\chi^2(2)$; where:

$$ \text{Mag}(\mu, \sigma) = \sum_{\gamma_{t+1}=0} \log(1 - \Phi((\Phi^{-1}(\alpha) - \mu)\sigma^{-1})) $$

$$ + \sum_{\gamma_{t+1} \neq 0} \{-(1/2)\log(2\pi\sigma^2) - (\gamma_{t+1} - \mu)^2(2\sigma^2)^{-1} - \log[\Phi((\Phi^{-1}(\alpha) - \mu)\sigma^{-1})]\}. $$

II.4 Generalized Method of Moment Duration-based Tests

A Generalized Method of Moment (GMM) Duration-based Tests proposed by Candelon et al. (2008) analyses VaR forecast tests as an expression of simple moment conditions, which can be tested within the well-known GMM framework. This test extends the framework proposed by Bontemps and Meddahi, (2005, 2006) and Bontemps (2006), and consists in using a GMM framework in order to test if durations of VaR violations are geometrically distributed. The objective is to choose the appropriated moments to know whether their empirical expectations are close to 0 or not.

A GMM Duration-based Conditional Coverage Test (Candelon et al., 2008)

For a given model and a fixed coverage rate $\alpha$, we consider a sequence of $N_I$ durations, denoted $d_i$ for $i = [1, \cdots, N_I]$, observed between two successive violations associated to the $\alpha\%$ VaR forecasts:

$$ d_i^\alpha = t_i^\alpha - t_{i-1}^\alpha \quad (22) $$

with $t_i^\alpha$ is the date of the $i^{th}$ violation for the level $\alpha$.

Under the conditional coverage assumption, the duration $d_i^\alpha$, for $i = [1, \cdots, N_I]$, are i.i.d. and has a geometrical distribution with a success probability equals to
the coverage rate $\alpha$. The null hypothesis of Conditional Coverage (CC) test can be expressed as:

$$
\begin{align*}
H_0 & : E[M(d_1^\alpha)] = 0 \\
H_1 & : E[M(d_1^\alpha)] \neq 0
\end{align*}
$$

where $M(d_1^\alpha)$ is a $(p \times 1)$ vector, whose components are the orthonormal polynomials of the $j$ moments, corresponding to the duration at the level $\alpha^6$. $p$ is the number of moment conditions ($p > 1$). $M(d_1^\alpha)$ components are denoted $M_j(d_1^\alpha)$, such as, for $j = [1, \ldots, p]$:

$$
M_{j+1}(d_1^\alpha) = [(1 - \alpha)(2j + 1) + \alpha(j - d_1^\alpha + 1)] [(j + 1)(1 - \alpha)^{1/2}]^{-1} M_j(d_1^\alpha)
$$

$$
- j(j + 1)^{-1} M_j-1(d_1^\alpha),
$$

with $M_1(d_1^\alpha) = 0$ and $M_0(d_1^\alpha) = 1$.

Under some regularity conditions, we have:

$$
\left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} M_j(d_1^\alpha) \right)^2 \xrightarrow{d} \chi^2(1), \quad (23)
$$

for $j = [1, \ldots, p]$.

In an i.i.d. context these moments are asymptotically independent with unit variance and the conditional coverage (CC) statistic test is:

$$
J_{CC}(p) = \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} M(d_1^\alpha) \right)^T \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} M(d_1^\alpha) \right) \xrightarrow{d} \chi^2(p), \quad (24)
$$

with $p$ the number of orthonormal polynomials used as moment conditions.

**A GMM Duration-based Unconditional Coverage Test** (Candelon et al., 2008)

A GMM Duration-based Unconditional Coverage Test is obtained as a special case of the conditional test by considering only the first orthonormal polynomial, i.e. when $M(d_1^\alpha) = M_1(d_1^\alpha)$. The statistic for Unconditional Coverage (UC) Test, denoted $J_{UC}$, is equivalent to $J_{CC}(p)$ with $p = 1$:

$$
J_{UC} = \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} M_1(d_1^\alpha) \right) \xrightarrow{d} \chi^2(1). \quad (25)
$$

**A GMM Duration-based Independence Test** (Candelon et al., 2008)

Candelon et al., 2008) propose a separate test for the independence hypothesis. It consists in testing the hypothesis of a geometric distribution (implying the absence of dependence) with a success probability equal $\pi$; where parameter $\pi$ can be either fixed a priori, either estimated and is not necessarily equal to the

---

6In the continuous case, it is well known that the Pearson family of distributions (Normal, Student, Gamma, Beta, Uniform, ...) can be associated to some particular orthonormal polynomials whose expectation is equal to zero.
coverage rate $\alpha$. The statistic for Independence (Ind) Test, denoted $J_{Ind}$, is defined such as:

$$J_{Ind}(p) = \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} M(d^p_i) \right)' \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} M(d^p_i) \right) \xrightarrow{d} N \rightarrow \infty \chi^2(p),$$

where $M(d^p_i)$ denotes a $(p \times 1)$ vector whose components are the orthonormal polynomials $M_j(d^p_i)$ for $j = [1, \ldots, p]$, evaluated for a success probability equals to $\pi$.

II.5 A Dynamic Quantile Test (Engle and Manganelli, 2004)

The Dynamic Quantile (DQ) test is proposed by Engle and Manganelli (2004) to detect a model for the VaR estimation. Indeed, this model should produce a sequence of unbiased and uncorrelated exception indicator variables. Under the null hypothesis, all coefficients in the regression (26) of the exception indicator variables on its past values and on current VaR estimate, as well as the intercepts are zero, such as:\(^7\)

$$I_t = \delta_0 + \sum_{k=1}^{5} \delta_k I_{t-k} + \delta_6 VaR_{t-1}(\alpha) + \epsilon_t,$$

were $I_t$ is the exception indicator variable and $\delta_i, i = [1, \ldots, 6]$, are real coefficients.

\(^7\)Following Ferreira and Lopez (2005), 5 lags are used in this paper.